

Notes for Instructors

This guide provides a few teaching points for the Activities in each Lesson. We hope to point out areas where students run into problems and some of the errors they fall into. Experienced teachers will already be aware of these pitfalls, but perhaps a preview will help newer instructors prepare for the inevitable.

We also list the objectives for the Lesson, some suggested Concept Questions, and Questions for Writing or Discussion. As their name implies, you can ask students to turn in written responses to these questions, or they can be used to initiate class discussion. Concept Questions are multiple choice and can be delivered via a personal response system ("clickers"), or you can issue each student a set of colored index cards for answering the questions. Alternatively, any of these questions can be used on quizzes or tests.

If you wish, class time can be spent primarily on the Activities. You may want to give a brief introduction, but should start on the Activities as soon as possible to allow students sufficient time. Usually, students should work on the Activities in groups of three or four, while you circulate to answer questions and provide support. For some Lessons it is more effective to lead the class as a whole in a guided inquiry. Allow time at the end of class for a summary and wrap-up.

Lesson 1.1 Linear Models

This is a longish Lesson that reviews some skills from algebra in the context of applications: solving simple equations and inequalities, evaluating expressions, and graphing an equation by plotting points. You may wish to budget more than one day for this section as students settle into the class and refresh their skills.

We consider linear models of the form $y = (\text{starting value}) + (\text{rate of change}) \cdot x$. We find the intercepts of a linear model, and interpret their meaning in the context of the model. Some students will have forgotten how to find the intercepts, and most will need help interpreting the meaning of the intercepts in context. A common mistake is to give the meaning of the variable on the appropriate axis, instead of its value at the intercept.

If some students need more help with their skills, you can refer them to the corresponding Lesson in the Algebra Toolkit.

We also review the intercept method for graphing a line. Finally, we consider linear models given in the general linear form, $Ax + By = C$. If you find that students' algebra skills are rusty, you might assign some appropriate review from the Algebra Toolkit for this Section.

You may notice that Activity 2 of the Lesson uses the same model as one of the Exercises (or Checkpoints) in Section 1.1 of the text. We occasionally duplicate exercises in this way in a not so subtle effort to encourage students to read the text.

Teaching Points for the Classroom Activities

It's a good idea to work through Lesson 1 for the class, so that they have a model to follow for future Lessons. You might use a tablet computer, a document camera, or overhead transparencies to show them exactly how to write their work on the page.

Activity 1

- Part (b): Point out that the terminology " L in terms of t " tells us which variable goes on which axis.
- Part (c): Graphs should extend across the entire grid, including any intercepts.
- Parts (d) and (e): Have students draw dashed lines on the graph, as in Example 1.2.
- Part (f): Emphasize the difference between evaluating an expression and solving an equation.

Activity 2

- Part (d): Teach students to start with $t = 16$, $C = 45$, then replace the variables with words, and finally write a sentence.
- Part (e): Highlight the appropriate portion of the graph, and then the corresponding portion of the horizontal axis.
- Part (g): Remind students how to write and solve a compound inequality.

Activity 3

- Part (b): You may have to remind students that t is measured from 2007, not from the previous table entry; so that for instance in 2015, $t = 8$ (not 5).
- Part (e): To interpret the C -intercept, for example, start with $C = 490$ when $t = 0$, then replace the variables with words, and finally write a sentence.

Activity 4

- Part (a): Many students will immediately write an equation (the answer to part (b)), but will have trouble answering the two questions in part (a).
- Part (e): Many students will not know what to do with the 5000.

Objectives for Lesson 1.1

- Analyze linear models given by a description in words, by a graph, by a table of values, or by an equation.
- Write an equation for a linear model described in words, in the form:
$$y = (\text{starting value}) + (\text{rate}) \times t$$
- Interpret an ordered pair of values for a linear model.
- Answer questions about a model by evaluating an expression or solving an equation or inequality.
- Make a graph from a table of values by choosing appropriate scales for the axes and plotting points.
- Use a graph to evaluate an expression or solve an equation.
- Find and interpret the intercepts of a linear graph.
- Graph a line by the intercept method.
- Write a linear model in general linear form.
- Solve a linear equation for y in terms of x .

Questions for Writing or Discussion

1. Describe some ways to display the relationship between two variables.
2. What is the difference between an expression and an equation?
3. What is the difference between evaluating an expression and solving an equation?
4. Explain how to find the x -value that corresponds to a given y -value on a graph.
5. Explain how to graph a line by the intercept method.
6. Explain how the words *intercept* and *intersect* are related, and how they are different.
7. In what situations is the general linear form easier to use for a model than the form $y = (\text{starting value}) + \text{rate} \times t$?
8. Explain why setting $y = 0$ in a linear equation allows us to find the x -intercept.

Concept Questions

1. How many points do you need to graph a linear equation?
 - a. Two
 - b. Three
 - c. One in each quadrant
 - d. It depends on the equation
2. When you graph the data given in a table, on which axis do you show the variable in the first row of the table?
 - a. The linear axis
 - b. The horizontal axis
 - c. The vertical axis
 - d. Both axes
3. If C is expressed in terms of H , which variable goes on the horizontal axis?
 - a. C
 - b. H
 - c. x
 - d. The smaller one
4. If $x > 5$, what is true about $-2x$?
 - a. It is greater than 3
 - b. It is less than 3
 - c. It is greater than 10
 - d. It is less than -10
5. Suppose a line is increasing and its y -intercept is positive. Which is true?
 - a. Its x -intercept is positive.
 - b. Its x -intercept is negative.
 - c. We can't tell.
 - d. The line may not have an x -intercept.
6. Is it possible for the x -intercept and the y -intercept of a line to be the same point?
 - a. No
 - b. Yes
 - c. Only for a vertical line
 - d. They are always the same point.

7. What is the y -coordinate of any point on the x -axis?
 - a. (x, y)
 - b. Y
 - c. It depends on the value of x .
 - d. 0
8. Delbert says that the intercepts of the line $3x + 5y = 30$ are $(10, 6)$. What is wrong with his answer?
 - a. $(10, 6)$ is not on the x -axis.
 - b. An intercept must have a 0 coordinate.
 - c. The line has two intercepts.
 - d. All of the above.

Lesson 1.2 Functions

In this Lesson we introduce the basic notation and terminology for functions. There will probably be some confusion about whether a table represents a function: is it allowed to have two different inputs with the same output? When discussing function notation, emphasize that in the equation $y = f(x)$, f is not a variable, $f(x)$ does not indicate a product, and $f(x)$ is the output variable, or another name for y .

Teaching Points for the Classroom Activities

Activity 1

- Students may have trouble filling in the table, so you might do another row or two with them.
- Check students' graphs before going over the remaining questions. A few arithmetic mistakes can make for very inaccurate graphs!
- The class usually finds the questions easy once their graphs are correct.

Activity 2

- In part (c), you may need to explain that there can be several ways to "estimate" an answer.

Activity 3

- To help them answer part (d), remind students to read the problem again.

Objectives for Lesson 1.2

- Recognizing functions
- Evaluating functions defined by tables, graphs, or equations
- Using function notation

Concept Questions

1. What distinguishes a function from other variable relationships?
 - a. The variables are related by a formula.
 - b. The values of the input and output variables must be different.
 - c. There cannot be two output values for a single input value.
 - d. There cannot be two input values for a single output value.
2. Use function notation to write the statement " L defines w as a function of p ."
 - a. $L = w(p)$
 - b. $w = L(p)$
 - c. $p = L(w)$
 - d. $L = p(w)$
3. Name three ways to describe a function.
 - a. By inputs, outputs, or evaluation
 - b. By tables, equations, or graphs
 - c. By the intercepts, the slope, or the vertex
 - d. By numbers, letters, or diagrams

4. If $n = f(a)$, what are the input and output variables?
 - a. f is the output and n is the input
 - b. a is the input and f is the output
 - c. a is the input and n is the output
 - d. $f(a)$ is the input and n is the output

Questions for Writing or Discussion

1. How would you know if a table of values does not come from a function?
2. What does it mean to evaluate a function?
3. Do linear equations $y = mx + b$ and quadratic equations $y = ax^2 + bx + c$ define functions? Why or why not?
4. Give an example of a function in which two distinct values of the independent variable correspond to the same value of the dependent variable.

Lesson 1.3 Graphs of Functions

In this Lesson we use function notation with graphs, construct the graph of a function, use the vertical line test, and solve equations and inequalities graphically. Some students find this last task very difficult, so we have included an auxiliary Lesson at the end of the guide.

Teaching Points for the Classroom Activities

Activities 1 and 2

- A major goal of these Activities is to help students distinguish between input and output values on a graph, or between x and $f(x)$, as in "what is the maximum value of the function" versus "where did the maximum occur."
- Students will probably be unfamiliar with the terminology "over what interval," so you will need to explain that this refers to input values.

Activity 3

- Most students still find solving inequalities graphically quite difficult. It may be helpful to go over several of the homework problems on this topic.

Objectives for Lesson 1.3

- Read function notation from a graph
- Interpret the coordinates of a point on the graph of a function
- Construct the graph of a function
- Use the vertical line test
- Solve an equation or inequality graphically

Concept Questions

1. If $f(8) = 2$, what point lies on the graph of f ?
 - a. $(8, 2)$
 - b. $(2, 8)$
 - c. Both (a) and (b).
 - d. All points between 2 and 8.
2. Write an equation that says that the point $(8, 17)$ lies on the graph of g .
 - a. $8 + 17 = g$
 - b. $g + 8 = 17$
 - c. $g(8) = 17$
 - d. $8 = \frac{17}{g}$

3. Which of the following is true?
 - a. It is not possible for the function $y = f(x)$ to take on the same y -value at two different x -values.
 - b. The maximum value of $y = f(x)$ may occur at two different x -values.
 - c. The maximum value of the function $y = f(x)$ is the largest x -value that appears on the graph.
 - d. $f(x) = 0$ at the y -intercept of $y = f(x)$.
4. When is a function called decreasing?
 - a. If its x -values increase from left to right.
 - b. If its y -values decrease when its x -values decrease.
 - c. If its y -values decrease when its x -values increase.
 - d. If the graph lies below the x -axis.

Questions for Writing or Discussion

1. Describe how to find the value of $f(3)$ from a graph of f .
2. Explain how to construct the graph of a function from its equation.
3. Explain how to use the vertical line test.
4. Explain how to solve an inequality by using a graph.

Lesson 1.4 Slope

This Lesson reviews the notion of slope, which students learned in elementary algebra, but emphasizes the meaning of the slope as a rate of change. Notice that in the beginning of this Lesson we do not use the coordinate formula for slope,

$m = \frac{y_2 - y_1}{x_2 - x_1}$; instead all of the slopes in the Lesson can be computed from the

definition, $m = \frac{\Delta y}{\Delta x}$, and we emphasize that approach. Encourage students to include units in their calculation of Δy and Δx to help them interpret the slope as a rate. The two-point formula is introduced at the end of the textbook section, and will be used in the next section to introduce the point-slope formula for a line.

Two skills will probably be new to students: deciding if data are linear by checking for constant slope, and using the slope to calculate either Δy or Δx , given a value for the other.

Teaching Points for the Classroom Activities

Activity 1

- Explain the difference between total change and average yearly change.
- In part (d), remind students that their answer should be a rate of change, with units.

Activity 2

- Many students will work out part (d) using "common sense." Show them how to use the definition of slope and solve an equation to get the same answer.

Activity 3

- Part (a) is intended to help students learn to work with Δx , Δy , and subscripted letters as ordinary variables.
- Part (b): many students have trouble understanding the difference between x and Δx .

Objectives for Lesson 1.4

- Use ratios and rates for comparison.
- Compute and interpret a rate of change.

- Interpret slope as a rate of change.
- Illustrate slope on a graph.
- Use the definition of slope to compute Δy or Δx .
- Recognize a linear relationship by its constant rate of change.

Concept Questions

- Suppose y gives the weight in pounds of a bottle of water that holds x quarts. What are the units of $\frac{\Delta y}{\Delta x}$?
 - Water per bottle
 - Quarts per pound
 - Pounds per quart
 - Quart-pounds
- What is wrong with the following reasoning? The point $(10, 7)$ lies on the graph of a line, so the slope of the line is $\frac{7}{10}$.
 - The slope is $\frac{10}{7}$.
 - 7 is not an intercept
 - We need two points to compute the slope.
 - The line is decreasing.
- How can you tell from the equation of a line whether its graph is increasing or decreasing?
 - Its slope is positive.
 - Its x -intercept is positive.
 - Its slope is negative.
 - Its y -intercept is negative.
- Comment on the following calculation: The intercepts of a line are $(0, 3)$ and $(5, 0)$, so the slope of the line is $\frac{3}{5}$.
 - It is true because we used two points.
 - It is false because Δx and Δy have opposite signs.
 - It is true because the values are increasing.
 - It is false because the slope is $\frac{5}{3}$.

Questions for Writing or Discussion

- What do we use ratios and rates for?
- Suppose x represents your distance from home, in miles. Compare the meanings of the two statements $x = 30$ and $\Delta x = 30$.
- Explain how slope measures the steepness of an incline.
- State two useful consequences of the fact that lines have constant slope.

Lesson 1.5 Linear Functions

In this Lesson, we review the slope-intercept formula for linear functions. It has already appeared in this course as the form for a linear model. Here we use it to graph a linear equation and to find the equation of a line from its graph. We also introduce the coordinate point-slope form and illustrate its use as a model, but explore it more fully in the next Lesson.

Teaching Points for the Classroom Activities

Activity 2

- To help students with part (a), ask: if $\frac{1}{8}$ mile costs \$0.30, how much would 1 mile cost? You can also point out that dividing \$0.30 by $\frac{1}{8}$ mile gives the same result as multiplying \$0.30 by 8.
- In part (b), remind students of the form for a linear model. In this model the input variable is distance (miles), not time.
- Students may find part (d) difficult: they don't understand that the models in part (b) are expressions for the taxi fare.

Activity 3

- Using variables other than x and y can be confusing for students. Write the formula $T - T_1 = m(S - S_1)$ directly below $y - y_1 = m(x - x_1)$, and sketch a set of axes labeled S and T next to axes labeled x and y .

Objectives for Lesson 1.5

- Identify the slope and y -intercept from the equation for a line
- Interpret the slope and y -intercept in context
- Use the coordinate formula to calculate slope
- Use the point-slope form as a linear model

Concept Questions

- What do the coefficients in the slope-intercept form tell you about a line?
 - m is the slope; b is the x -intercept
 - m is the slope; b is the y -intercept
 - (m, b) is a point on the line
 - m is the x -intercept; b is the y -intercept
- What will be wrong with your answer if you accidentally compute the slope as $m = \frac{y_2 - y_1}{x_1 - x_2}$?
 - The number will be too big.
 - The line will be decreasing.
 - That is the slope of the perpendicular line.
 - It will have the wrong sign.
- What do you get when you substitute the point $(0, b)$ into the point-slope formula?
 - $x = 0$
 - $y = mx + b$
 - $y = b$
 - $ax + by = 0$
- What is the easiest way to find the slope of the line $18x - 42y = 60$?
 - Solve for y to get the slope-intercept form.
 - Find the intercepts and use them to compute the slope.
 - Graph the line and compute $\frac{\Delta y}{\Delta x}$.
 - Find values of x and y that make the equation true.

Questions for Writing or Discussion

- Explain why the initial value for a linear model is often given by the y -intercept.
- How can you find the equation of a line from its graph?
- How is the coordinate formula for slope related to net change?
- Explain the difference between the slope-intercept form and the point-slope form for a linear equation.

Lesson 1.6 Linear Regression

This Lesson begins with the point-slope formula. Besides being able to use the formula, we want students to understand two ideas:

- The point-slope formula is really the same as the coordinate formula for slope, but used in a different way.
- We'd also like students to understand the syntax of the formula, i. e. starting at a point (x_1, y_1) and using the slope to find any other point on the line, rather than just memorizing it.

Students may have been taught to find a linear equation by substituting a point into the slope-intercept formula to find b , but at this stage it is important that they master the point-slope formula. Finding a linear equation to fit two points is one of the most useful skills they will learn in this course. Note that we give two alternate forms for the formula: the fractional form, $\frac{y - y_1}{x - x_1} = m$, is easier for students if the slope is a common fraction. The form $y = y_1 + m(x - x_1)$ captures the mechanics of the formula, and will occur in calculus.

This Lesson also provides a brief introduction to linear regression. We begin by considering data that are "almost" linear and fit a regression line by eye. Explain to students that their answers may differ slightly; nonetheless a line can be a bad fit. Show them how to adjust their line so that the same number of data points lie above the line and below the line. You will also need to emphasize that the line of best fit need not pass through any of the data points, and that to find the equation of the regression line, they must use points on the line, not data points.

One goal of finding a line of best fit is for students to practice using the point-slope formula. In an application, the hardest part for students is identifying the two points to use. Encourage them to make a short table, starting by deciding which is the input variable and which is the output.

Teaching Points for the Classroom Activities

Activity 1

- Use a clear plastic ruler to draw a line of best fit.
- Ask students to choose two points that lie on the regression line but are not data points. Look for points that lie as close as possible to grid lines, so that their coordinates are easy to read.
- Work through part (c), finding the equation of the regression line. Emphasize a two-step process: find the slope, then apply the point-slope formula.

Activity 2

- You might enter the data ahead of time, so that you can help students as they enter the data.
- Students will probably notice that the regression line appears to pass through only one data point. Emphasize again that it need not pass through any data points.

Objectives for Lesson 1.6

- Find a linear model from two data points
- Sketch a line of best fit for a scatterplot
- Find an equation for a line of best fit
- Make estimates using interpolation and extrapolation
- Use technology for least-squares regression

Concept Questions

1. Can two points on a scatterplot have the same x -coordinate? Can two points have the same y -coordinate?
 - a. Yes, Yes
 - b. Yes, No
 - c. No, Yes
 - d. No, No
2. If you add more data points to the scatterplot, could the regression line change?
 - a. Yes
 - b. No

3. How many data points must a good regression line pass through?
 - a. Three
 - b. Two
 - c. One
 - d. None
4. We can estimate the regression equation by drawing the line that passes through:
 - a. The first and last data points
 - b. Two data points in the middle
 - c. Either of these
 - d. None of these

Questions for Writing or Discussion

1. Describe a strategy for sketching a line of best fit by eye.
2. Should you expect the data points on a scatterplot to satisfy the regression equation? Why or why not?
3. Explain the difference between interpolation and extrapolation.
4. Explain how to find an equation for a line through two points using two steps.

Lesson 2.1 Nonlinear Models

At this point, it may be useful to review solving an equation by graphing (in particular a quadratic equation), and to stress the difference between an exact solution and a decimal approximation. Point out that our first nonlinear models include the Pythagorean theorem, formulas for volume and surface area, and the compound interest formula.

Teaching Points for the Classroom Activities

We recommend going over these Activities with the class, using a document camera or overhead projector. Almost all of the skills required will be new to most students. Even students who recently completed an algebra course have trouble with extraction of roots: although solving $x^2 = 16$ is easy for them, solving $(x + 3)^2 = 16$ is not. In particular, watch out for the mistake of solving $A = P(1 + r)^2$ by subtracting P from both sides.

Activities 1 and 2

- Help students find the appropriate formulas in the textbook
- Students may also need help entering their functions into their calculators or graphing utility

Activity 3

- If there is time, present examples of solving other simple nonlinear equations

Objectives for Lesson 2.1

- Solve quadratic equations and formulas by extraction of roots.
- Understand the difference between an exact value and an approximation
- Use models involving quadratic equations
- Solve other simple nonlinear equations

Concept Questions

1. Which of the following equations cannot be solved by extraction of roots?
 - a. $2x^2 + 3 = 8$
 - b. $2x^2 + 3x = 8$
 - c. $2(x + 3)^2 = 8$
 - d. $2x^2 - 3 = 0$

2. Which solutions are approximations?
 - a. $x^2 = 40$, $x = \pm 6.32455532$
 - b. $t^2 = \frac{81}{64}$, $t = \pm 1.125$
 - c. $w^2 = 50$, $w = \pm 5\sqrt{2}$
 - d. $b^2 = (0.632)^2$, $b = \pm 0.632$
3. What is the first step in solving the equation $2(x + 1)^2 = 17$?
 - a. Expand $(x + 1)^2$
 - b. Get zero on one side
 - c. Divide both sides by 2
 - d. Take the square root of both sides
4. Which statement is true?
 - a. A quadratic equation may include a linear or a constant term.
 - b. The solutions of a quadratic equation are always of the form $\pm k$.
 - c. Your calculator gives exact decimal values for square roots of integers.
 - d. The coefficients of a quadratic equation are called parabolas.

Questions for Writing or Discussion

1. Use graphs to explain why a linear equation can have only one solution, but a quadratic equation may have two solutions.
2. Explain the difference between the volume and the surface area of a cylinder. Include units in your explanation.
3. What is the difference between a rational number and an irrational number? Give examples of each.
4. Delbert squared -6 by entering -6^2 into his calculator, and got -36 . What went wrong?

Lesson 2.2 Some Basic Graphs

In this lesson we review absolute value and study the graphs of eight basic functions: $y = x$, $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $y = |x|$. When considering the two rational functions, we introduce the notions of vertical and horizontal asymptotes. Note that these topics should be treated intuitively and numerically; in particular we do not explicitly discuss limits. Nor do we yet use the terminology of domain and range. At this stage, students still think of functions and graphs pointwise, and will only gradually progress to a more global view.

After doing a few examples of simplifying expressions involving cube roots and absolute values, let the class work in groups on the Activities. You might display examples of students' graphs and discuss their strengths and weaknesses.

Teaching Points for the Classroom Activities

Activity 1

- It is helpful to graph each pair of functions together on the graphing calculator, and compare the two graphs in different windows. In particular, look at the window $[0, 1] \times [0, 1]$ and observe that the graphs intersect at $(1, 1)$.

Activity 2

- Here is our first encounter with asymptotes. It may be helpful to graph a few translations of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ on the calculator, e.g. $y = \frac{1}{(x - 2)}$ and $y = \frac{1}{x^2} + 3$, to show that asymptotes do not always coincide with the axes.

Objectives for Lesson 2.2

- Evaluate cube roots and absolute values
- Graph eight basic functions
- Use horizontal and vertical asymptotes for graphing
- Graph functions defined piecewise

Concept Questions

1. Which of the eight basic functions are undefined at $x = 0$? (Choose all that apply.)
a. $y = x$ b. $y = x^2$ c. $y = x^3$ d. $y = \sqrt{x}$
e. $y = \sqrt[3]{x}$ f. $y = \frac{1}{x}$ g. $y = \frac{1}{x^2}$ h. $y = |x|$
2. Which of the eight basic functions are undefined for negative x ? (Choose all that apply.)
a. $y = x$ b. $y = x^2$ c. $y = x^3$ d. $y = \sqrt{x}$
e. $y = \sqrt[3]{x}$ f. $y = \frac{1}{x}$ g. $y = \frac{1}{x^2}$ h. $y = |x|$
3. Which of the eight basic functions are always positive? (Choose all that apply.)
a. $y = x$ b. $y = x^2$ c. $y = x^3$ d. $y = \sqrt{x}$
e. $y = \sqrt[3]{x}$ f. $y = \frac{1}{x}$ g. $y = \frac{1}{x^2}$ h. $y = |x|$
4. If $|x| = -x$, what can you say about x ?
a. x must be zero.
b. x must be negative.
c. x must be zero or negative.
d. This cannot happen for any value of x .

Questions for Writing or Discussion

1. Describe the differences between the graphs of $f(x) = x^2$ and $g(x) = x^3$.
2. Describe how we use a vertical asymptote to help us sketch a graph.
3. Compare the graphs of $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$.
4. What connections do you see between the graphs of $y = x^3$ and $y = \sqrt[3]{x}$?

Lesson 2.3 Transformations of Graphs

For this Lesson, it is probably more efficient if you display the graphs for the class. Students who are more comfortable with their graphing utility may wish to follow along, and all students can write the answers to the questions in their Activity booklets.

Teaching Points for the Classroom Activities

Activity 2

- Students usually find vertical translations easy to grasp, but horizontal translations are harder. You might refer them to the tables in the textbook, or point out that if you pull the x -axis to the right, the graph appears to move to the left.

Activity 3

- For stretches and compressions, be sure to consider examples other than $y = x^2$, because a parabola squashed vertically looks very much like a parabola stretched horizontally.
- Part of the purpose of this Lesson is to help students become familiar with the eight basic graphs.

Objectives for Lesson 2.3

- Recognize and sketch transformations of the basic graphs
- Write a formula for a transformation of a graph
- Interpret transformations of graphs in context

Concept Questions

1. The y -intercept of the graph of $y = f(x)$ is $(0, 2)$. What point lies on the graph of $y = f(x + 3)$?
 - a. $(0, 5)$
 - b. $(3, 2)$
 - c. $(-3, 2)$
 - d. $(3, 5)$
2. The graph of $y = g(x)$ has a vertical asymptote at $x = -4$. What happens to the asymptote under a vertical translation?
 - a. Nothing
 - b. It is compressed vertically
 - c. It is translated vertically
 - d. It is eliminated
3. The graph of $y = F(x)$ is symmetric about the y -axis. Which of the following graphs will also be symmetric about the y -axis?
 - a. $y = -3F(x)$
 - b. $y = F(x) - 3$
 - c. $y = F(x - 3)$
 - d. (a) and (b)
4. Under which transformations does the graph of a function keep the same shape?
 - a. Vertical translation
 - b. Horizontal translation
 - c. Reflection
 - d. All of these

Questions for Writing or Discussion

1. How would you alter the formula for $y = f(x)$ if you wanted each y -coordinate on the graph to be $\frac{1}{3}$ as far from the x -axis?
2. Delbert says that the graph of $y = f(x + 5)$ adds 5 units to the x -coordinate of each point on the graph of $y = f(x)$. Is he correct? Why or why not?
3. How do you think the graph of $y = f(-x)$ looks, compared to the graph of $y = f(x)$?
4. Francine applies a vertical translation to her graph, and then applies a vertical compression to the result. Would she get the same graph if she applied the transformations in the opposite order?

Lesson 2.4 Functions as Mathematical Models

In this lesson students are asked to match the verbal description of a model with an appropriate formula and one of the basic graphs. We discuss the shape of a graph, particularly its concavity, as it applies to modeling.

Teaching Points for the Classroom Activities

Begin the class with a brief discussion of concavity and changing slopes. The graphs in this Activity show only the first quadrant, so students must look more closely at their shapes to choose the appropriate graph for each formula. While reading the descriptions, they will also be reviewing the language of proportionality.

Activity 1

- Students may need help distinguishing between the graphs of $y = x^2$ and $y = x^3$, between $y = \sqrt{x}$ and $y = \sqrt[3]{x}$, and between $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$.

- The first model uses the familiar formula $d = rt$ (or $t = \frac{d}{r}$), but in the other models you may need to remind students how to find k from the values given.
- They can answer part (c) using either the formula or the graph, and can perhaps use one to verify the other.

Objectives for Lesson 2.4

- Sketch a graph whose shape models a situation
- Choose one of the basic graphs to fit a situation or a set of data
- Decide whether the graph of a function is increasing or decreasing, concave up or concave down

Concept Questions

1. A graph whose slopes decrease for increasing x is called
 - a. increasing.
 - b. decreasing.
 - c. concave up.
 - d. concave down.
2. Which graph is concave down wherever it is defined?
 - a. $y = x^3$
 - b. $y = \frac{1}{x}$
 - c. $y = \sqrt[3]{x}$
 - d. $y = \sqrt{x}$
3. To decide which basic function might model a set of data, we can
 - a. plot the data
 - b. look at the variables
 - c. choose scales for the axes
 - d. calculate the slope
4. A piecewise function
 - a. must have no breaks in its graph.
 - b. can have more than one output at some inputs.
 - c. cannot have concavity.
 - d. none of the above

Questions for Writing or Discussion

1. Describe and compare the graphs of the eight basic functions, including intervals where the graphs are positive or negative, increasing or decreasing, and concave up or down.
2. Explain the difference between an empirical model and a mechanistic model.
3. How can you tell from a table of values whether a function is concave up or concave down?
4. What does an open circle on the graph of a piecewise function mean?

Lesson 2.5 The Absolute Value Function

In this lesson we solve absolute value equations and inequalities, and calculate error tolerance.

Teaching Points for the Classroom Activities

Give a short lesson on using the notation $|x - a| = b$ to discuss distance. If you wish to cover measurement error, you could do that now.

The Activities for this Lesson are devoted to solving absolute value equations and inequalities.

Activity 1

- We solve absolute value equations three ways: with a graph, with a number line, and algebraically. Point out that when using a graph, the solution appears on the x -axis, and looks the same as the solution obtained on a number line.
- In Part 3(b), get students to see that we can replace $-(x - 3) = 5$ by $x - 3 = -5$. This will be useful when we solve inequalities.

Activity 2

- When choosing the correct intervals on the x -axis, use the same language we used when solving quadratic inequalities: are the correct x -values "between" the boundary points, or "outside" of them?
- Students at this level usually find it difficult to follow an algebraic argument to obtain the conclusions on absolute value inequalities summarized in the text. It is more effective to rely on the graphical interpretation described above.

Objectives for Lesson 2.5

- Use absolute value notation to write statements about distance
- Use graphs to solve absolute value equations and inequalities
- Solve absolute value equations and inequalities algebraically
- Express error tolerances using absolute value notation

Concept Questions

1. To find the absolute value of a number, we can
 - a. change its sign.
 - b. subtract it from 0.
 - c. find its distance from 0.
 - d. square it.
2. The notation $|x - 3| = 5$ means
 - a. x is 3 units bigger than 5
 - b. the distance between x and 5 is 3 units
 - c. 5 and 3 are x units apart
 - d. the distance between x and 3 is 5 units
3. Which statement is true?
 - a. The graph of $y = |2x - 8|$ has no negative inputs.
 - b. The equation $|2x - 8| = -4$ has two solutions.
 - c. Depending on x , $|2x - 8|$ can equal $2x - 8$ or $8 - 2x$.
 - d. The graph of $y = |2x - 8|$ is a straight line.
4. Which statement is false?
 - a. The equation $|3x + 12| = 0$ has one solution.
 - b. The statement $|T - 0.1| < 0.05$ describes an error tolerance of 0.05 units.
 - c. $|mx + b| < k$ is equivalent to $-k < mx + b < k$.
 - d. The solutions of $|3x + 12| \geq 6$ form a closed interval.

Questions for Writing or Discussion

1. Describe the absolute value graph.
2. Explain how to solve an absolute value inequality with a graph.
3. Find the error in this statement and correct it:
"If $|x - 3| > 5$, then either $x - 3 > 5$ or $x - 3 > -5$."
4. What is error tolerance, and how is it used?

Lesson 2.6 Domain and Range

This Lesson contains several geometry problems that lend themselves to a discussion of domain and range. Working through these problems will provide good practice with algebraic skills.

Teaching Points for the Classroom Activities

If students struggle with the first Activity, you might work through it with them, and perhaps assign Activity 2 as homework. Note that the function in Activity 2 is a rational function, and the function in Activity 3 is a square root.

Activity 1

- Draw a few different lines that pass through $(2, 1)$ for students.
- Students may have trouble finding a and b in terms of m . Remind them that the intercepts are points with coordinates $(a, 0)$ and $(0, b)$.

Activity 2

- Note that the setting for this Activity is the same as the previous one, but using a different independent variable.
- In each Activity, encourage students to use both algebraic and graphical methods to find the domain and range of the function.

Objectives for Lesson 2.6

- Find the domain and range of a function from its formula
- Find the domain and range of a function from its graph
- Find the range of function on a given domain

Concept Questions

1. Which function changes concavity on its domain?
 - a. $y = x^2$
 - b. $y = x^3$
 - c. $y = \sqrt{x}$
 - d. $y = |x|$
2. Which function includes all real numbers in its range?
 - a. $y = x^2$
 - b. $y = x^3$
 - c. $y = \sqrt{x}$
 - d. $y = |x|$
3. In a function, how many domain elements can be paired with one range element?
 - a. One
 - b. Two
 - c. None
 - d. Infinitely many
4. The domain of the function $f(x)$ is $[-4, 4]$. What is the domain of the function $g(x) = \frac{1}{2}f(x)$?
 - a. $[-2, 2]$
 - b. $[-8, 8]$
 - c. $[-4, 4]$
 - d. $[4, -4]$

Questions for Writing or Discussion

1. In Activity 1, Delbert came up with the equation $\frac{b}{a} = \frac{1}{a-2}$. How did he get this?
2. Francine says that the expression $\sqrt{36 - m^2}$ can be simplified to $6 - m$. Is she correct? Why or why not?
3. Do the largest and smallest values in the domain of a function result in the largest and smallest values of the range? Give an example.
4. Which of the eight basic functions include all real numbers in their range?

Lesson 3.1 Variation

In this lesson we consider direct and inverse variation, as a precursor to the later study of power functions. We consider how to identify different forms of variation from a table, a graph, or an equation, and how to model applications with variation equations.

Teaching Points for the Classroom Activities

As you work through the Activities, keep a record on the blackboard listing the properties for the three main types of variation: direct variation, direct variation with a power (for this course, the power is almost always two or at most three), and inverse variation. Include how to recognize from a table of values, the form of the equation, and the shape of the graph.

Depending on the ability of your class, you may not be able to include a discussion of scaling and scale "factors."

Activity 1

- Point out that the constant of variation is just the constant T/p , which is also the slope of the graph.
- Go over the general strategy for variation problems, emphasizing how to decide which sort of variation is described.

Activity 2

- In part (2), point out that a slight discrepancy in the value of k can be explained by round-off error.
- In part (4), values of the variables are not given, but can be read from the graph.

Activity 3

- Point out that not every decreasing function represents inverse variation; we must test for $xy = k$.
- In direct variation, the graph passes through the origin, but in inverse variation the function is undefined at $x = 0$. The graph has both a vertical and a horizontal asymptote.

Objectives for Lesson 3.1

- Recognize direct and inverse variation from a table of values
- Find the constant of variation and write an equation
- Sketch the graph of a direct or inverse variation
- Use scaling in inverse or direct variation

Concept Questions

1. When does a table represent direct variation?
 - a. If it has a constant slope
 - b. If it includes the point $(0, 0)$
 - c. If the ratio output/input is constant
 - d. If each output is double the previous one
2. If y varies directly with x , what does the constant of variation tell us?
 - a. The y -intercept of the graph
 - b. The slope of the graph
 - c. What happens to y when you double x
 - d. The size of the input values

3. How can you test whether a table for $y = f(x)$ represents direct variation with a power?
 - a. Plot the points
 - b. Check whether y^n is a constant
 - c. Check whether y/x^n is a constant
 - d. Check whether y^n/x^n is a constant
4. How can you test whether a table for $y = f(x)$ represents inverse variation?
 - a. Check whether xy is constant
 - b. Check whether the function is decreasing
 - c. Check whether y is the reciprocal of x
 - d. Check whether y/x is constant

Questions for Writing or Discussion

1. Explain why the function $f(x) = 4x + 1$ is not an example of direct variation.
2. Describe the scaling property of direct variation.
3. Describe the graph of an inverse variation.
4. Give examples of applications that exhibit direct variation, direct variation with a power, and inverse variation.

Lesson 3.2 Integer Exponents

In this lesson we define negative and zero exponents, revisit the laws of exponents, write numbers in scientific notation, and use power functions in applications.

Teaching Points for the Classroom Activities

A review lesson on the laws of exponents appears before Lesson 6.1 in the Activities booklet. You may wish to assign these problems as homework, or to review selected problems in class.

Begin class with a brief review of negative exponents and how they relate to power functions, leading into Activity 1.

Activity 1

- After completing parts 3(a) and (b), ask students which basic graph they expect the graph of $m = f(d)$ to resemble.
- In part 3(e), many students will have difficulty solving for d when d^2 appears in the denominator. Dividing both sides of $300 = \frac{k}{d^2}$ by k and getting d^2 on the right side is a common error.
- Use part (4) to discuss changing from one unit of measure to another; in this case, from watts to picowatts. You should also make sure that students can read scientific notation on their calculators, and stress that "2.8 E -6", for example, is not an acceptable form for an answer.

Activity 2

- Do parts 1-4 verbally with the class, stopping to explain any that generate questions.
- In part 5, show students how to substitute values for a and b .

Objectives for Lesson 3.2

- Evaluate negative exponents
- Write a formula for a power function
- Graph and analyze power functions

- Solve equations involving negative exponents
- Simplify expressions using the laws of exponents

Concept Questions

- Which of these is equivalent to $2x^{-4}$?
 - $-2x^4$
 - $\frac{1}{2x^4}$
 - $2 - x^4$
 - $\frac{2}{x^4}$
- Which statement is true about power functions?
 - They can describe direct or inverse variation.
 - They involve a power of the output variable.
 - The y -intercept must be a positive number.
 - They include all linear and quadratic functions.
- Which of the following is not one of the laws of exponents?
 - $(x + y)^n = x^n + y^n$
 - $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
 - $(xy)^n = x^n y^n$
 - $x^n x^m = x^{n+m}$
- Which of these is equivalent to $\frac{m^2}{m^6}$?
 - $\frac{1}{m^3}$
 - $-m^4$
 - $\frac{1}{m^4}$
 - $\frac{1}{3}$

Questions for Writing or Discussion

- What is an inverse square law? Give an example.
- Explain why $\frac{1}{a^{-n}} = a^n$, if $a \neq 0$.
- How can you tell whether a number should have a positive or a negative exponent in scientific notation? Explain why this is true.
- Give a numerical example to show why we cannot add or subtract terms with the same variable but different exponents.

Lesson 3.3 Roots and Radicals

In this lesson we define powers of the form $a^{1/n}$, use the laws of exponents on such powers, and convert expressions from radical to exponential form and vice versa. (Note that we do not yet consider powers with exponents m/n ; those come in the next lesson.) We briefly discuss irrational numbers and the roots of real numbers. Finally, we consider equations and power functions that involve $a^{1/n}$.

After introducing exponents of the form $\frac{1}{n}$, you might want to go over most of the Exercises in the text as a skills warm-up. Emphasize the difference between negative exponents and fractional exponents. Show students how to evaluate n th roots on their calculators using both radicals and exponents.

Teaching Points for the Classroom Activities

Activity 1

- In part (a), tell students that we are not using the third row of the table (v_c) until later in the problem.
- In part (b), make sure that students extend their graphs to pass through the origin, and that they don't try to fit a straight line through the points.

Activity 2

- Ask students whether they expect smaller animals to have slower or faster heart rates than larger ones, and what that means for the function $H = f(m)$. Then ask them how they know that $H = km^{-1/4}$ is a decreasing function.
- In part (d) we solve an equation. You may want to illustrate the process with one or two more examples, for example, Exercise 9 from the text.

Activity 3

- Ask students to complete the box in part 1, but go over parts 2 and 3 with the class.
- In part 4, explain that the statements are or are not applications of the fourth law of exponents, but most students will have trouble until they grasp that the formulas "work" for multiplication, but not for addition or subtraction.

Objectives for Lesson 3.3

- Evaluate fractional powers and roots
- Convert between radical and exponential notation
- Graph and analyze power functions
- Solve equations involving radicals
- Simplify expressions using the laws of exponents

Concept Questions

1. Which of the following expressions is not equivalent to the other three?
a. $B^{1/8}$ b. $\sqrt[8]{B}$ c. $B^{0.125}$ d. $\frac{1}{B^8}$
2. Which of the following is undefined for negative x ?
a. $x^{1/4}$ b. x^{-4} c. $(-x)^{1/4}$ d. $(-x)^{-4}$
3. Which of these do we accomplish by taking a fifth root?
a. Solve $\sqrt[5]{x} = 32$ b. Evaluate $f(x) = x^{-5}$ for $x = 32$
c. Solve $x^5 = 32$ d. Solve $x^{1/5} = 32$
4. Which of the following statements is true?
a. $x^{1/2} + x^{1/2} = x$ b. $x^{1/4} \cdot x^4 = x$
c. $x^{-1/5} = \frac{-1}{x^5}$ d. $x^{1/3} \cdot x^{1/3} \cdot x^{1/3} = x$

Questions for Writing or Discussion

1. Explain why $x^{1/4}$ is a reasonable notation for $\sqrt[4]{x}$.
2. How many real-valued roots does a real number have?
3. Compare the graphs of $y = x^{1/n}$, for $n = 2, 3$, and 4.
4. Explain the difference between a rational and an irrational number. Give examples.

Lesson 3.4 Rational Exponents

In this lesson we extend the study of rational exponents to powers of the form $a^{m/n}$. We simplify expressions using the laws of exponents, solve equations, and look at models that use power functions with rational exponents.

Begin class by reviewing the types of exponents we have already studied, and show how the definition of rational exponents make sense. You may want to practice using the laws of exponents, but do not make the examples unreasonably complicated, or the discussion may devolve into a lesson on adding fractions. It is also useful to solve a few equations that involve rational exponents.

Teaching Points for the Classroom Activities

Activity 1

- Part (a) involves working with scientific notation. Show students how to simplify the formula for a with minimal calculator use.
- You might want to do part (d) in two ways, using the original equation and the answer to part (a).

Activity 2

- Consider assigning this Activity as a project or for extra credit.

Objectives for Lesson 3.4

- Evaluate powers with rational exponents
- Convert between radical and exponential notation
- Graph and analyze power functions with rational exponents
- Solve equations involving rational exponents
- Simplify expressions using the laws of exponents

Concept Questions

1. $x^{3/4}$ means
 - a. the fourth root of x cubed
 - b. the cube root of x to the fourth
 - c. the $3/4$ root of x
 - d. the cube root of the fourth root of x
2. To solve the equation $x^{2/3} = k$, we should
 - a. Raise both sides to the negative of the exponent
 - b. Divide both sides by the exponent
 - c. Raise both sides to the reciprocal of the exponent
 - d. Raise the right side to the given exponent
3. Which of the following is the correct way to evaluate $15^{3/5}$ on a calculator?
 - a. $15^3 \div 5$
 - b. $15 \div 5^3$
 - c. $(15 \div 5)^3$
 - d. $15^{(3 \div 5)}$
4. The notation $a^{0.6}$ means
 - a. $\sqrt[6]{a}$
 - b. $\frac{1}{a^6}$
 - c. $a^{1/6}$
 - d. $a^{3/5}$

Questions for Writing or Discussion

1. Explain how to compute $625^{3/4}$ by hand.
2. Explain how to solve $x^{0.4} = 18$, and why this method works.
3. Explain why $x\sqrt{x} = x^{1.5}$.
4. Suppose $M = 8 \times 10^6$. Explain why $M^{2/3} = 4 \times 10^4$.

Lesson 3.5 Joint Variation

This lesson gives two examples of joint variation and one instance of Simpson's paradox.

Activity 1

- In parts (c) and (d), you may need to remind students how to find a constant of variation.

Activity 3

- Here is another Simpson's paradox problem:

A college is being investigated for discrimination against women in its acceptance of students in the two graduate programs. Records show that a much higher percentage of the men who applied for graduate school were accepted than the corresponding percentage of women who applied. The table shows the number of men and women candidates who were accepted or denied admission into either the English or Engineering program.

	English Department		Engineering Department	
	Men	Women	Men	Women
Accepted	5	17	63	5
Denied	20	62	7	0

- Determine the total number of men who applied to graduate school at this college and the percentage of men who were accepted.
- Determine the total number of women who applied to graduate school at this college and the percentage of women who were accepted.
- Of the men who applied to the English program, what percentage were accepted?
- Of the women who applied to the English program, what percentage were accepted?
- Of the men who applied to the Engineering program, what percentage were accepted?
- Of the women who applied to the Engineering program, what percentage were accepted?
- Did men or women have a better success rate in each department? Which group had a better success rate overall?

Objectives for Lesson 3.5

- Use function notation for a function of two variables
- Find a formula for joint variation
- Analyze joint variation by holding one variable constant

Concept Questions

- Which is true about a function of two variables?
 - It has one input and two outputs.
 - It has two inputs and one output.
 - It has two inputs and two outputs.
 - None of the above.
- We can represent a function of two variables by
 - a table of values.
 - a graph in three dimensions.
 - several graphs in two dimensions.
 - all of the above.
- Which of the following functions represents joint variation, where k is a constant?
 - $H = kst$
 - $L = m + kn$
 - $Y = kR^2$
 - $Z = k(x + y)$

Questions for Writing or Discussion

- Explain how to make a table of values for a function of two variables.
- Look up the definition of Simpson's paradox, and explain how it applies to Activity 3.

Lesson 4.1 Exponential Growth and Decay

In this lesson we introduce exponential growth and decay. We do not define exponential functions as such until Lesson 4.2. Here we look at examples of growth and decay by a constant factor or a constant percent rate, and compare exponential with linear growth.

The Activities in the lesson are designed to introduce the new material, so you may want to begin class with the Activities and summarize after each part, instead of using an introductory lecture.

Teaching Points for the Classroom Activities

Activity 1

- To help students write a function for the population, show them how the function values in the table are built up with powers of 3, and lead them to see that the exponent on $P(t)$ is t .
- Show the class how to evaluate exponential expressions on the calculator.
- After completing part 1, go over the formula for exponential growth, $P(t) = P_0 b^t$, and point out that we only need to know the initial value and the growth factor to write an equation. Use this observation in part 2.

Activity 2

- Students usually have some difficulty grasping the formula for percent increase. You may have to explain several times "where the 1 comes from" in $1 + r$: it gives us back the "old" value, while r produces the increase.
- In part 2, if students persist in writing $b = 0.06$ instead of $b = 1.06$, you might ask them to create a short table for their version of the formula, and see if it is increasing.

Activity 3

- Notice that in part 1, 0.90 is the value of $1 - r$, not of r . After completing part 1, you will need to emphasize the distinction between decreasing to 90% (of the earlier value) and decreasing by 90%. You may want to state several examples with different wordings and different values of r for students to write formulas.
- Also in part 1, point out that we don't see much of a curve to the graph over just 5 decades, but if we change the graphing window to 10 or 15 decades, the exponential shape is more pronounced.

Activity 4

- After getting students started (point out that only the populations in 1992 and 2000 are relevant to the problem), you might assign the write-up to hand in.

Objectives for Lesson 4.1

- Calculate percent increase or decrease
- Write a formula for exponential growth or decay
- Evaluate an exponential growth or decay function
- Find the growth factor or initial value
- Solve for percent increase or decrease
- Compare linear and exponential growth

Concept Questions

1. A population grows by 4.5% per year. What is its annual growth factor?
a. 4.5 b. 0.45 c. 0.045 d. 1.045
2. A population decreases by 12% per year. What is its annual decay factor?
a. -0.12 b. -1.12 c. 0.88 d. 1.88

3. The function $P(t) = P_0 b^t$ is decreasing if
 - a. P_0 is negative b. b is negative
 - c. $0 < b < 1$ d. $0 < P_0 < 1$
4. The graph of $P(t) = P_0 b^t$, where $P_0 > 0$ and $b > 0$, is always
 - a. increasing b. concave up
 - c. decreasing d. concave down

Questions for Writing or Discussion

1. How can you tell from a table of values that a function is increasing or decreasing exponentially?
2. If an exponential function increases by 500 in the first month, will it also increase by 500 in the second month? Why or why not?
3. Explain why the growth factor for an exponential function is $1 + r$, where r is given by a percent increase, but the decay factor for a decrease given by percent is $1 - r$.
4. If an exponentially decreasing function decreased by 500 this month, will it decrease by more than 500 or less than 500 next month? Why?
5. If you know only two points on a graph, can you determine whether the graph is linear or exponential? Explain.

Lesson 4.2 Exponential Functions

In this lesson we introduce the formal definition of exponential function. Explain to students why we make the restrictions $b > 0$ and $b \neq 1$; trying to evaluate $f(x) = (-4)^{1/2}$ is a good example. Note that the input values can be positive or negative, but output values are always positive. Have students graph several examples on their calculators.

We also introduce exponential equations. Note that in this first treatment, we do not use logarithms; we solve only equations in which both sides can be written with the same base. The purpose of this approach is to emphasize that we are looking for an unknown exponent on a known base.

Teaching Points for the Classroom Activities

Activity 1

- Show students how to enter exponential formulas into the calculator, and demonstrate that $4\left(\frac{1}{2}\right)^x$ is not the same as 2^x .
- You may need to give several more examples comparing exponential and power functions.

Activity 2

- Point out that it is usually easiest to write both sides in the lower of the two possible bases.
- As you work examples, emphasize the statement in part (b), and that we are not really "cancelling" the bases.
- Students often have difficulty applying the laws of exponents to simplify expressions.
- In part (2), this is the first example in which a population triples every 4 days, instead of every day. Look at several examples to help students understand that the formula for the population should be $P_0(3)^{t/4}$ instead of $P_0(3)^t$.

Activity 3

- In the table for part (2), demonstrate again that the ratio of equally spaced function values is constant, not their difference.

- The tables in part (3) provide another opportunity to consider the difference between linear and exponential functions. Students will need to be reminded of the "equal ratios" test throughout this chapter, as they tend to persist in believing that all functions share the properties of linear functions.

Objectives for Lesson 4.2

- Graph exponential functions
- Distinguish exponential functions from power functions and other functions
- Solve exponential equations with the same base
- Solve exponential equations graphically

Concept Questions

- Which statement is true?
 - An exponential function is not defined for negative inputs.
 - The outputs of an exponential function cannot be negative.
 - The y -intercept of the function $f(x) = 2(3)^x$ is $(0, 6)$.
 - The function $g(x) = 16(0.5)^x$ decreases by 8 each time we increase x by 1.
- Which of the following is an exponential function?
 - $f(x) = 3x^4$
 - $f(x) = 3(4)^x$
 - $f(x) = 2x^{3/4}$
 - $f(x) = \frac{4}{x^3}$
- Which is a good strategy for solving $3^{x-2} = 81$?
 - Divide both sides by 3.
 - Add 3^2 to both sides.
 - Simplify the left side.
 - Write the right side as a power of 3.
- If g is an exponential function, what is the last entry in the table?

x	0	1	2
$g(x)$	48	36	?

- 24
- 27
- 12
- 18

Questions for Writing or Discussion

- What is the difference between a power function and an exponential function?
- How can you recognize an exponential function from a table of values?
- If a population grows by 6% annually, why is its growth factor 1.06?
- If a population decreases exponentially, does it lose the same number of people each year? Why or why not?

Lesson 4.3 Logarithms

In this lesson we introduce logarithms as a tool for solving exponential equations. Note that we do not yet consider logarithmic functions. Instead, we make an analogy between using radicals to solve power equations and using logarithms to solve exponential equations. You might want to begin class with the problem posed in the reading assignment.

Teaching Points for the Classroom Activities

Activity 1

- Students are often confused because the symbol for logarithm, "log" looks like a word, instead of a mathematical symbol like $\sqrt{}$. Explain that $\log_2 x$ means "find an exponent on base 2 that gives x ," just as $\sqrt{x}$ means "find a number that when

squared gives x ." It may help to do several examples mentally before turning to the calculator.

- Explain that the log button on the calculator finds exponents on base 10 that aren't easy to guess, just as the square root button finds square roots that aren't easy to guess.
- Students will probably ask how to use the calculator to find logs with other bases. Tell them that that information is coming later, and for now any logs that aren't base 10 will be easy enough to guess.

Activity 2

- Part (1) defines a logarithm as an unknown exponent on a given base.
- In part (2) we reinforce the definition of logarithm as an exponent by converting equations from logarithmic form to exponential form. In each example, emphasize that "the logarithm is the exponent."

Activity 3

- The notion that taking a logarithm is the inverse operation for exponential functions is quite hard for students at this point. They may need time to understand the identity $\log_b(b^x) = x$.
- In part (e), it may help to list powers of 10 on the board and locate numbers between powers of 10.

Activity 4

- Now we can solve exponential equations with base 10. Once again, students may ask how to solve equations with other bases, and you can tell them that's coming up in the next lesson.
- From their work with power functions, most students will have learned to start solving by "isolating the power." However, you will have to explain several times why taking the log of both sides eliminates the base. Try to steer them away from saying that we "cancel" the base, instead using language such as "undoing" the exponential function.
- Help students enter the steps of the solution into their calculators, reminding them frequently not to round off intermediate results.
- In part (3), emphasize the difference between part (b), where we evaluate the exponential function, and part (c), where we solve an exponential equation. Point out that many application problems are really asking for one or the other of these operations.

Objectives for Lesson 4.3

- Compute logarithms using the definition
- Convert between exponential and logarithmic form
- Solve exponential equations with base 10
- Approximate logarithms graphically

Concept Questions

1. A logarithm is the same as
 - a. an exponent
 - b. a coefficient
 - c. a quotient
 - d. a radical
2. $\log_b b = 1$ because
 - a. $\frac{b}{b} = 1$
 - b. $\sqrt[b]{b} = 1$
 - c. $b^1 = b$
 - d. $1^b = 1$

3. If $\log x = 2.5$, what can you say about x ?
 - a. $x = 25$
 - b. x is between 20 and 30
 - c. x is between 100 and 1000
 - d. $x = \frac{1}{2.5} = 0.4$
4. What is the first step in solving the equation $5(10)^x = 12$?
 - a. Multiply 5 times 10.
 - b. Take the log of both sides.
 - c. Get zero on one side of the equation.
 - d. Divide both sides by 5.

Questions for Writing or Discussion

1. Explain how to estimate $\log_{10} x$ between two integers.
2. Explain how to use a graph of $y = 3^x$ to estimate $\log_3 12$.
3. Explain why $\log_b 1 = 0$.
4. If $\log_2 x$ is negative, what can you say about x ?

Lesson 4.4 Properties of Logarithms

In this lesson we introduce the properties of logarithms and use them to solve exponential equations. Instead of trying to "prove" the properties, it may be more effective to illustrate them with numerical values, as is done in the reading. Remind students that " $\log(x)$ " is function notation: the parentheses do not indicate multiplication, and are thus not subject to the distributive law. When solving equations, you will also need to emphasize that $\frac{\log a}{\log b} \neq \log\left(\frac{a}{b}\right)$. Students often have a hard time applying a general rule expressed algebraically to specific instances, so it is a good idea to work through several examples.

We use a two-step strategy for solving the exponential equations we study: we isolate the exponential expression, then take the log of both sides.

Teaching Points for the Classroom Activities

Activity 1

- To help your students understand the log properties, look at some numerical examples, as in part (a), alongside the corresponding property of exponents. Highlight the fact that the logarithms are the exponents.
- Point out the various incorrect applications of the log properties that students are prone to make.

Activity 2

- When solving exponential equations, students may try to evaluate $\frac{\log a}{\log b}$ as $\log\left(\frac{a}{b}\right)$, either by misapplying the log property, or by failing to use parentheses on the calculator.

Activity 3

- This Activity uses an exponential model to review the two skills of evaluating a function and solving an equation, and deciding which is appropriate.

Objectives for Lesson 4.4

- Use the properties of logarithms to simplify expressions
- Solve exponential equations using base 10 logarithms
- Solve formulas involving exponential expressions

Concept Questions

- Which statement is true?
 - $\log(8 + y) = \log 8 + \log y$
 - $\log(8y) = \log 8 + \log y$
 - $\log(8y) = \log 8 \cdot \log y$
 - $\log\left(\frac{8}{y}\right) = \frac{\log 8}{\log y}$
- Which expression is equivalent to $3\log_2 H - \log_2 T$?
 - $3\log_2(H - T)$
 - $\log_2(H^3 - T)$
 - $\log_2 \frac{H^3}{T}$
 - $\log_2\left(\frac{H}{T}\right)^3$
- How can we solve $4^x = 60$?
 - Divide both sides by 4.
 - Take log base 10 of both sides.
 - Take the fourth root of 60.
 - This equation has no solution.
- Which formula gives the amount when interest is compounded quarterly?
 - $P(1 + 4r)^t$
 - $P(1 + 4r)^{4t}$
 - $P\left(1 + \frac{r}{4}\right)^{4t}$
 - $P\left(1 + \frac{r}{4}\right)^{t/4}$

Questions for Writing or Discussion

- Explain why the distributive law does not apply to the expression $\log(a + b)$.
- In the compound interest formula, what do the expressions $\frac{r}{n}$ and nt stand for?
- How can you calculate $x = \log_3 12$? Hint: first convert to exponential form.
- How does Property (3) of logarithms help us solve exponential equations?

Lesson 4.5 Exponential Models

In this Lesson we fit an exponential model to two data points, and we consider doubling time and half-life.

Teaching Points for the Classroom Activities

Try to get students to understand this fundamental difference between linear and exponential functions: while linear functions increase or decrease by equal amounts over equal time intervals, exponential functions increase or decrease by equal factors over equal time intervals. If possible, it is also desirable for students to understand that there is nothing special about the factor 2; it is merely convenient.

At this point in their study of exponential functions, students begin to confuse the various parameters. Take a few minutes to review some basic definitions: $P(t)$ is a variable, but P_0 is a constant; r is the percent growth rate and $b = 1 + r$ is the growth factor; and so on.

Activity 1

- In part (1), round the doubling time to 9 years to simplify the sketch in part (d)
- For part (2), you may need to explain that "quarterly" means four time per year, and so on.

Activity 2

- Use the graph to emphasize that the substance is not completely gone after two half-lives!
- Students usually need to see several examples before they can convert easily between decay rate and half-life, or between growth rate and doubling time.

Activity 3

- In part (a), check that students substitute the coordinates into $f(t) = ab^t$.
- Emphasize that this gives a system of equations, but not a linear system.
- You might point out that if students round their value of b , the formula will not yield the given function values exactly.

Objectives for Lesson 4.5

- Calculate the doubling time or half-life of an exponential function
- Write exponential functions using doubling time and half-life
- Find an exponential function given two points

Concept Questions

1. You have written a system of equations to fit an exponential function through two points. What is the next step?
 - a. Calculate the slope.
 - b. Subtract one equation from the other.
 - c. Divide one equation by the other.
 - d. Take the log of both sides.
2. Which statement is true?
 - a. The doubling time of a population depends on its initial value.
 - b. An increasing exponential function has a constant doubling time.
 - c. The doubling time is twice the percent growth rate.
 - d. The doubling time is half the percent growth rate.
3. The formula $P(t) = P_0 2^{t/D}$ tells us that:
 - a. $P(t)$ is twice the initial value.
 - b. After 2 years, $P(t) = D$.
 - c. After D years, $P(t) = 2P_0$.
 - d. $P(t)$ grows by a factor of D .
4. The half-life of DDT is 15 years. This means that:
 - a. 30 pounds of DDT dissolve in one year.
 - b. 100 pounds of DDT dissolve in 30 years.
 - c. After 30 years, 100 pounds of DDT is reduced to 25 pounds.
 - d. Each half-pound of DDT takes 15 years to dissolve.

Questions for Writing or Discussion

1. Explain why you don't need to know the initial amount of a radioactive substance in order to calculate its half-life.
2. Explain how to use the half-life to sketch the graph of an exponential function.
3. Explain how to find the growth factor for an exponential function when you know its doubling time.
4. Does an exponential function triple in equal amounts of time? Explain.

Lesson 5.1 Inverse Functions

In this Lesson we study inverse functions and their graphs and apply the horizontal line test to determine whether the inverse is itself a function.

Teaching Points for the Classroom Activities

Inverse functions are often somewhat nebulous for students. Try to make the analogy with inverse operations, that a function (at least the ones we are considering) tells you to perform some operation on the input. The Activities in this Lesson attempt to demystify inverse functions by looking at their graphs and treating them like any other familiar function.

Activity 1

- Students can sketch the inverse graphs either by interchanging the coordinates of points on the graph or by working out the formula for the inverse first. They can use one method to check the other.
- You might have each group sketch one graph, then display the results to the class.
- They should, of course, notice the relationship between $y = x^2$ and $y = \sqrt{x}$, and between $y = x^3$ and $y = \sqrt[3]{x}$. You could point out that they already knew that, for example, taking square roots is the "inverse operation" for squaring.

Activity 2

- Walk around to make sure that students have their initial graphs correct.
- They may have trouble finding the formula for $g^{-1}(x)$ in part (2); give them a hint to get them started.
- Ask students if they see a relationship between the graphs of the function and its inverse. Reflection about the line $y = x$ may be difficult for them to see without help.

Activity 3

- Students usually have no trouble applying the horizontal line test. There will be some controversy over part (1d); it is impossible to tell from the picture whether the graph has asymptotes or vertical lines.

Activity 4

- Part (1) is intended to help students see that the exponential and log functions are not mysterious, but similar to other familiar functions.
- You may need to provide an example for Part (3). An answer for (a) might be

$$\begin{aligned}f(f^{-1}(8)) &= (\sqrt[3]{8})^3 = 2^3 = 8 \\f^{-1}(f(x)) &= \sqrt[3]{2^3} = \sqrt[3]{8} = 2\end{aligned}$$

Objectives for Lesson 5.1

- Find a formula for an inverse function
- Graph an inverse function
- Apply the horizontal line test
- Find domain and range of an inverse function

Concept Questions

1. If the point $(3, -4)$ lies on the graph of f , what point lies on the graph of its inverse function?
 - a. $(-3, 4)$
 - b. $(-4, 3)$
 - c. $(4, -3)$
 - d. $\left(\frac{1}{3}, \frac{-1}{4}\right)$
2. The graphs of f and f^{-1} are symmetric about
 - a. the x -axis
 - b. the y -axis
 - c. the asymptote
 - d. the line $y = x$

3. An inverse function
 - a. makes the output negative
 - b. makes the input negative
 - c. interchanges the input and output
 - d. takes the reciprocal of the output
4. Which of these is the inverse of the function $h = m - 2$?
 - a. $h = m + 2$
 - b. $h = 2 - m$
 - c. $h = \frac{1}{m} - 2$
 - d. $h = \frac{1}{m - 2}$

Questions for Writing or Discussion

1. If the function f tells you how much money is in your account after t years, what does the function f^{-1} tell you?
2. Explain the difference between the meaning of the notation p^{-1} if p is a function or if p is a variable.
3. Which of the basic functions is its own inverse? Why?
4. What does the term one-to-one mean? Give an example.

Lesson 5.2 Logarithmic Functions

In Chapter 4 we treated the logarithm as an operation; taking a log was the inverse operation for raising to a power, just as taking a square root is the inverse operation for squaring. Now we investigate the log function as the inverse function for the exponential function. We also solve logarithmic equations using two steps: we use the log properties to combine the logs into one log, then convert to exponential form.

Teaching Points for the Classroom Activities

Applications of log functions are not nearly as common as applications of exponential functions, so most students will find that thinking of logarithms as an operation is more useful than using a log function, but the added practice provided by this Lesson will probably be helpful.

Activity 1

- Hick's Law is an interesting logarithmic model from psychology, along with the Weber-Fechner Law. Why logs base 2? Psychologists hypothesize that the mind divides a list successively into halves.

Activity 2

- The graphs are very close; students will have to be precise in their work. After they have finished their graphs, it might be instructive to display a large version using Desmos or some other graphing facility.
- You might also point out that log graphs (eventually) grow very slowly, slower than any root. (Because, of course, exponential functions eventually grow faster than any power.)

Activity 3

- We solve simple log equations by converting to exponential form. You may need to review the equivalence with some numerical examples.
- For more complicated equations, stress that the first step is to combine the logs into one log, using the log properties.

Objectives for Lesson 5.2

- Convert expressions from exponential to log form, and vice versa
- Graph a log function
- Evaluate a log function
- Solve a log equation

Concept Questions

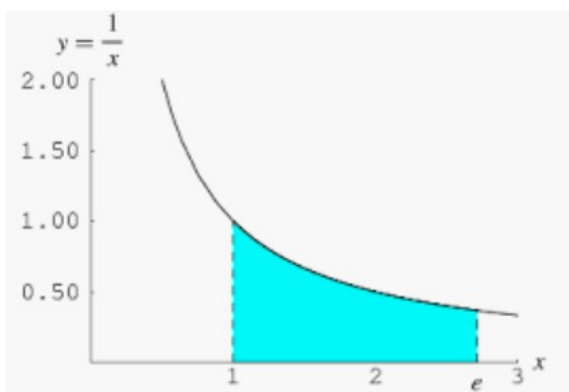
1. What is the y -intercept of the graph of $f(x) = \log_5 x$?
 - a. $(0, 1)$
 - b. $(1, 0)$
 - c. $(0, 5)$
 - d. there is none
2. The domain of the function $g(x) = \log_3 x$ is
 - a. all real numbers
 - b. all multiples of 3
 - c. all non-negative numbers
 - d. all positive numbers
3. Which of these is the first step in solving the equation: $\log x + \log(x - 1) = 2$?
 - a. $\log(2x - 1) = 2$
 - b. $\log x = 2 - \log(x - 1)$
 - c. $\log x(x - 1) = 2$
 - d. $x + x - 1 = 2^x$
4. Which statement is true?
 - a. The log of a number is never negative.
 - b. We cannot take the log of a negative number.
 - c. The log of a fraction is called a common log.
 - d. The log of 0 is 1.

Questions for Writing or Discussion

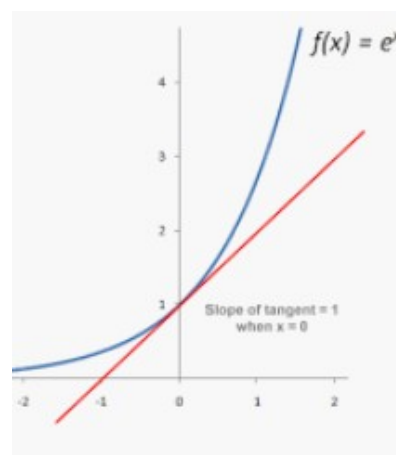
1. Explain why we cannot take the log of a negative number.
2. Why does the function $f(x) = \log x$ grow so slowly?
3. Compare the graphs of $f(x) = \log\left(\frac{1}{x}\right)$ and $g(x) = -\log x$, and explain.
4. What is an extraneous solution?

Lesson 5.3 The Natural Base

It is difficult to explain to students before calculus why the natural base is important. This Lesson includes two interesting numerical facts about e , but we have not attempted to justify its relevance to exponential functions. If you wish, you might offer the two usual pictures to the class:



and



Teaching Points for the Classroom Activities

Activity 1

- Having students graph by hand may help them accept $y = e^x$ as just another exponential function.
- To reinforce the idea, you might ask them to calculate 2.7^x as well as, or instead of, using the calculator's "black box" e^x key.
- In part (2), they should graph the inverse functions by interchanging the coordinates of points on the exponential graphs.

Activity 2

- In part (2), it might be helpful to have students begin by making a table of values for $n!$ for n up to 6.

Activity 3

- This Activity deals with change of base. Part (1) is covered in the text: writing an exponential function with base e .
- Part (2) demonstrates the change of base formula for logarithms.
- Keep reminding students that "ln" is ell-en, for natural log, and not Eye-en, for inverse or something else.

Objectives for Lesson 5.3

- Evaluate the natural exponential and natural log functions
- Solve exponential equations base e
- Model exponential growth and decay with base e

Concept Questions

1. The value of e^2 is closest to
 - a. 3
 - b. 5
 - c. 7
 - d. 9
2. Which of the following is equivalent to $e^x = k$?
 - a. $e^k = x$
 - b. $\log_k e = x$
 - c. $\log_e k = x$
 - d. $k^x = e$
3. Why is the equation $e^x = 6.5$ easier to solve than $8^k = 6.5$?
 - a. 8 is larger than 6.5
 - b. k is a constant
 - c. There is a button for log base e on the calculator, but not a button for log base 8.
 - d. Because e is an irrational number.
4. The natural log of a fraction between 0 and 1 is
 - a. positive
 - b. negative
 - c. undefined
 - d. between e^0 and e^1

Questions for Writing or Discussion

1. Explain how to rewrite $P(t) = P_0 b^t$ with the natural base.
2. Delbert says that he will begin solving the equation $100e^{0.6t} = 40$ by computing $e^{0.6}$. Is this a good strategy? Why or why not?
3. Explain why $\ln\left(\frac{1}{e^3}\right) = -3$.
4. Explain why solving exponential equations in base e is no harder than solving exponential equations in base 10.

Lesson 5.4 Logarithmic Scales

In this Lesson we look at logarithmic scales and their application to pH, decibels, and earthquake magnitude. Students should be able to plot and read values on log scales labeled with powers of 10 or with integer values. They should understand that a difference of k units on a log scale corresponds to an increase in magnitude by a factor of 10^k .

Teaching Points for the Classroom Activities

Activity 1

- In part (b), have the class use the table feature on the calculator to evaluate $\log x$ for $x = 0.06, 0.6, 6, 60, 600$, and 6000 . To reinforce the conclusion, you might list powers of ten and their logs on the board.
- In part (c), point out at the start that the value of A_0 will not be needed in the calculations.

Objectives for Lesson 5.4

- Plot and read values on a logarithmic scale
- Solve problems involving log scales such as pH, decibels, and the Richter scale

Concept Questions

1. What is a log scale used for?
 - a. To find the logarithm of a number.
 - b. To plot data that covers a wide range of values.
 - c. To highlight the curvature in the graph of an exponential function.
 - d. To convert logarithms to base 10.
2. A value of 5682.7 would be plotted between which two integers on a log scale?
 - a. 5682 and 5683
 - b. 5000 and 6000
 - c. 5 and 6
 - d. 3 and 4
3. How much stronger is magnitude 4 earthquake than a magnitude 2 earthquake?
 - a. Twice as strong.
 - b. Four times as strong.
 - c. 16 times as strong.
 - d. 100 times as strong.
4. Which statement is false?
 - a. We use log scales to graph a variable that has a wide range of values.
 - b. On a log scale, we actually plot exponents.
 - c. Values less than one appear as negative numbers on a log scale.
 - d. Equal increments on a log scale correspond to equal differences in value.

Questions for Writing or Discussion

1. How does a log scale differ from a linear scale?
2. If $B = 100A$, the difference between A and B on a log scale is 2 units. Use the properties of logarithms to explain why this is true.
3. Explain what negative values on a log scale mean.
4. How much stronger is a magnitude 6 earthquake than a magnitude 3 earthquake? Explain.

Lesson 6.1 Intercepts and Factors

Students find this Lesson quite difficult; in fact, you may need to spend two sessions on the lesson, one session on intercepts, factors and solutions, and one session on application problems.

Begin by pointing out that $ax^2 + bx + c$ is an expression, which we may be able to factor, $ax^2 + bx + c = 0$ is an equation, which has solutions, and $y = ax^2 + bx + c$ is an equation in two variables, which we can graph.

Some students will not remember how to factor quadratic expressions; if so you will probably have to review factoring outside of class time. And although most students remember the zero-factor principle, they often do not apply it correctly. You will also need to spend some time discussing whether to use factoring or extraction of roots to solve a given equation. It is a good idea to keep quizzing students on their solving skills as we progress to each new solution technique.

Although the Activities do not include finding a quadratic equation with given solutions, you should do some examples like Example 5 in class. Note that most students do not know what integers are.

Teaching Points for the Classroom Activities

Activity 1

- In the given window, the trace feature will not locate the x -intercept 3.5 exactly. Ask the class how they can verify algebraically that 3.5 is indeed the intercept.

Activity 2

- The point of part 1 is to show that we can factor a constant from a quadratic equation without changing its solutions. If students are not convinced that we can "ignore" the constant, you might point out that the equations $x - 2 = 0$ and $5x - 10 = 0$ have the same solution.
- In part 2, explain to the class that one of the main reasons they learn how to factor is to find the solutions of quadratic equations. However, if we know the solutions of the equation, we can write it in factored form. Show students how to use the Zinteger setting on the calculator.

Activity 3

- (This Activity also appeared in the Introduction to Chapter 2.)
- The graph of area versus base is quite abstract for students. Illustrate the concept by drawing several rectangles with perimeter 36 but different dimensions.
- Students also find it difficult to compute the height of the rectangle when its base is known. Do several numerical examples.
- Emphasize the fact that the graph shows how the area changes as we change the base of the rectangle; the height of the rectangle does not appear in this graph.

Activity 4

- This Activity solves a typical area problem three ways: by using a table (trial and error), by reading a graph, and by solving an equation. You will find that some students try to solve all future problems by trial and error, so be sure to assign some problems where the solutions are hard to "guess"!
- Once again, students have trouble writing algebraic expressions for the length and the area of the pasture.

Objectives for Lesson 6.1

- Solve quadratic equations by factoring.
- Find a quadratic equation with given solutions.

- Find the x -intercepts of a parabola.
- Solve problems involving perimeter and area or the Pythagorean theorem.

Concept Questions

- What are the x -intercepts of the graph of $y = 3(2x - 7)(x + 2)$?
 - $\frac{7}{2}$ and -2
 - $-\frac{7}{2}$ and 2
 - $3, \frac{7}{2}$ and -2
 - $3, -\frac{7}{2}$ and 2
- What happens to the x -intercepts when you multiply the right side of $y = ax^2 + bx + c$ by 3?
 - They are tripled.
 - They are divided by 3.
 - They move 3 units to the right.
 - They are unchanged.
- If the perimeter of a rectangle is 56 inches and its width is x inches, what is an expression for its length?
 - $56 - x$
 - $28 - x$
 - $56x$
 - $\frac{28}{x}$
- Which statement is true?
 - All rectangles with the same perimeter have the same area.
 - The solutions of $x(18 - x) = 80$ are $x = 80$ and $x = 18$.
 - If the perimeter of a rectangle is 20 cm, the largest area it can have is 20 sq cm.
 - If you know the x -intercepts of the graph of $y = x^2 + bx + c$, you can write it in factored form.

Questions for Writing or Discussion

- How can you use a graph to factor a quadratic expression?
- How are the factors of $ax^2 + bx + c$ related to the x -intercepts of the graph of $y = ax^2 + bx + c$?
- Explain why the solutions of $(x - 3)(x - 6) = 10$ are not 3 and 6.
- Explain why we cannot "cancel" $(x - 5)$ from both sides of the equation $3x(x - 5) = 6(x - 5)$. What are the solutions of the equation?

Lesson 6.2 Solving Quadratic Equations

This is a skills Lesson on completing the square and the quadratic formula. You may want to begin by presenting a quadratic equation that cannot be solved by factoring. To complete the square, our goal is to turn any quadratic expression into a square of a binomial, so that we can then solve the equation by extraction of roots. It is also helpful to tell students that although the quadratic formula is usually faster than completing the square, we'll need this skill for other applications (conic sections) later on.

Teaching Points for the Classroom Activities

Activity 1

- Have students square several binomials and look for the pattern in the coefficients before presenting the formula for $(x + p)^2$. Leave the formula prominently displayed on the board; you will need to refer back to it often.
- Some students catch on quickly, but others have a hard time finding p and p^2 . In particular, you will probably have to explain that if you know $2p$, you can find p by multiplying by $\frac{1}{2}$ (and this is the same as dividing by 2.)
- It may also help to write several quadratic trinomials on the board and ask students which are perfect squares (and then explain how you know.)

Activity 2

- In part 2, be sure to go over finding decimal approximations on the calculator.
- Forgetting to divide by the coefficient of x^2 is a common mistake.

Activity 3

- Try to eradicate the erroneous version of the quadratic formula, $x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$.
- Also point out the distinction between a quadratic equation and the quadratic formula.
- Note that we do not treat complex numbers yet, other than to acknowledge their existence. (Some students already know about complex numbers, so it is best to mention i briefly, but postpone further discussion for later.)

Activity 5

- This is an optional Activity. You can decide whether your students will benefit from it.

Objectives for Lesson 6.2

- Add a constant to $x^2 + bx$ to create a perfect square
- Solve quadratic equations by completing the square
- Use the quadratic formula to solve quadratic equations
- Solve quadratic equations for one variable in terms of the others

Concept Questions

1. What is the linear term of $(x + 6)^2$?
 - a. x^2
 - b. $6x$
 - c. $12x$
 - d. 36
2. What is the first step in solving $3x^2 - 6x = 2$?
 - a. Divide -6 by 2
 - b. Get zero on one side
 - c. Divide both sides by 3
 - d. Factor the left side
3. What should we add to $x^2 - 5x$ to create a perfect square?
 - a. 25
 - b. -25
 - c. $\frac{25}{4}$
 - d. 10
4. When is $(x + p)^2 = x^2 + p^2$?
 - a. Always
 - b. Never
 - c. When $x = p$
 - d. When $p = 0$
5. What is wrong with this statement of the quadratic formula? $x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
 - a. Nothing
 - b. We should simplify the radical
 - c. $-b$ should be over $2a$
 - d. It should be $\pm b$
6. Does $\sqrt{b^2 - 4ac} = b - \sqrt{4ac}$? Why or why not?
 - a. No, it should be $b - 2\sqrt{ac}$
 - b. Yes, we take the square root of each term
 - c. No, we cannot take the square root of a negative number
 - d. No, $\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$

Questions for Writing or Discussion

1. Explain how to tell whether $x^2 + bx + c$ is the square of a binomial.
2. How can you decide which of the three methods for solving a quadratic equation you should use?
3. How does creating the square of a binomial help us solve a quadratic equation?
4. Explain in words how to evaluate the quadratic formula.

Lesson 6.3 Graphing Parabolas

In this Lesson we consider the effects of the parameters a , b , and c on the graph of the basic parabola, discuss the discriminant and the number of x -intercepts of the graph, and consider more examples of the standard applications of quadratic models.

Teaching Points for the Classroom Activities

Activity 1

- Students have trouble evaluating even these simple quadratic expressions. A common error, for example, is to enter -4^2 into the calculator when evaluating x^2 at $x = -4$.
- After each part, summarize the results for students in part (c).
- After part 2, show the class how to find the x -intercepts algebraically. Note the symmetry of the graph and the location of the vertex.
- In part 3, again show students how to find the x -intercepts, pointing out the use of factoring instead of extracting roots to solve the equations.
- Show the class how to find the x -coordinate of the vertex as the average of the x -intercepts. They will also need to be prompted on how to find the y -coordinate of the vertex.

Activity 2

- This Activity leads students to the formula for the x -coordinate of the vertex. After deriving the formula, demonstrate that it works on the graphs in Activity 1, part 3.

Activity 3

- This Activity provides a short review and more practice with applications.

Objectives for Lesson 6.3

- Describe the effect of each of the parameters a , b , and c on the graph of a parabola
- Find the x - and y -intercepts of a parabola
- Use the discriminant to determine the number of x -intercepts
- Find the vertex of a parabola
- Sketch the graph of a parabola

Concept Questions

1. Which point on a parabola always lies on the axis of symmetry?
 - a. The x -intercept
 - b. The y -intercept
 - c. The vertex
 - d. The origin
2. The y -intercept of $y = ax^2 + bx + c$ is the same as the value of which parameter?
 - a. a
 - b. b
 - c. c
 - d. None of these
3. What does the value of a tell us about the graph of $y = ax^2 + bx + c$?
 - a. The y -intercept
 - b. The number of x -intercepts
 - c. The x -coordinate of the vertex
 - d. The width of the parabola
4. Which of these is the equation of a parabola that has no x -intercepts?
 - a. $y = x^2$
 - b. $y = x^2 + 4$
 - c. $y = x^2 - 4$
 - d. $y = x^2 + x$

Questions for Writing or Discussion

1. Describe what the parameter a tells you about the graph of $y = ax^2$.
2. Explain why the x -coordinate of the vertex is the average of the x -intercepts of the graph.

- Describe any differences between the graphs of $y = x^2 - 4$ and $y = x^2 - 4x$.
- Explain what the discriminant tells us about a quadratic equation.

Lesson 6.4 Problem Solving

In this Lesson we find the maximum or minimum value for a quadratic model and use the vertex form to find the equation of a parabola. We also consider systems in which one or both of the equations is quadratic.

Teaching Points for the Classroom Activities

Activity 1

- Have students graph the parabola in the "friendly" window $X_{\min} = -9.4$, $X_{\max} = 9.4$ to verify their answers.
- Try to get students to understand why the vertex in part 1 occurs at $(2, -6)$.
- Explain that many parabolas have the same vertex, so we need to know one other point to establish the "width" of the graph.

Activity 2

- Suggest to students that they graph the equation on the calculator, using the grid given to choose a window.

Activity 3

- This is one of several area and perimeter problems for students to practice with.
- In part (c), deciding which value corresponds to each variable is difficult for students.

Activity 4

- One or both of Activities 4 and 5 should be considered optional, depending on time constraints and the strength of your students.
- Stress that there are several unknown quantities in this problem, but all are expressed in terms of x , the number of price reductions.
- In part (f), you will need to remind students what x represents, and that they have expressions for the other unknown quantities.

Objectives for Lesson 6.4

- Find the maximum or minimum value of a quadratic model
- Write a quadratic equation in vertex form
- Find an equation for a parabola given the vertex and one other point
- Solve problems arising from quadratic models
- Solve a system of quadratic equations

Concept Questions

- What is the x -coordinate of the vertex of the parabola $y = 4(x - 2)(x + 8)$?
 - 4
 - 16
 - 6
 - 3
- What is the smallest y -value on the graph of $y = 4(x - 6)^2 + 12$?
 - 6
 - 4
 - 12
 - We can't tell without graphing
- The height of golf ball in meters is given by $h = -4.9t^2 + 20t$. How can you find out when the golf ball hits the ground?
 - Plug in $t = 0$
 - Solve for t when $h = 0$
 - Find the vertex of the graph
 - Calculate the discriminant
- What does the vertex of the graph tell us about the golf ball in question 3?

- How long the golf ball is in the air
- The starting height of the golf ball
- When the golf ball reaches its maximum height
- The speed of the golf ball

Questions for Writing or Discussion

- Explain how to find the coordinates of the vertex of a parabola.
- When is the vertex form for a quadratic equation useful?
- Explain why revenue will probably not increase indefinitely as price increases.
- Why do we need to know a second point besides the vertex to find the equation of a parabola?

Lesson 6.5 Quadratic Inequalities

In this Lesson we solve quadratic inequalities graphically and algebraically, and write their solutions with interval notation. Note that our algebraic technique uses properties of the parabola; we do not use a "sign graph" to solve inequalities.

Teaching Points for the Classroom Activities

Activity 1

- You might want to preface this Activity by having students use trial and error to find solutions to $x^2 - 4 \leq 0$, and plot their solutions on a number line on the board. (Remind them to try both positive and negative values, and fractions.) Then do the same for $x^2 - 4 > 0$.
- In part (b), explain that because the graph shows points for which $y = x^2 - 4$, the solutions to $x^2 - 4 < 0$ are just the x -coordinates of points on the graph with $y < 4$. You will need to go over this reasoning on every problem.
- In part (c), emphasize that because the solution set consists of two intervals, we need two inequalities to describe it. Also explain why $-2 > x > 2$ is not correct.
- In problem 2, start using the language "between the x -intercepts or outside the x -intercepts" or something similar to describe the choice of which part of the x -axis provides the solutions.
- In problem 3, we first rewrite the inequality with zero on one side. This problem also provides an opportunity to review graphing a parabola by hand. Notice that the coefficient of x^2 is negative, so expect some confusion about finding the x -intercepts versus finding the y -coordinate of the vertex.

Activity 2

- Remind students that once we have found the x -intercepts as boundary points for our solution set, we only need to decide whether the solutions lie between the intercepts or outside them -- we don't need any other information about the parabola (such as the vertex). So we only need to know if the parabola opens up or down.
- After problem 1, take a few minutes to give examples of interval notation. In this Lesson, we use $\cup$ to indicate the union of two intervals, but we do not discuss intersection of sets.
- Problems 3 and 4 point out that sometimes we have to use extraction of roots or the quadratic formula to find the x -intercepts.

Objectives for Lesson 6.5

- Write compound inequalities
- Use interval notation
- Solve quadratic inequalities graphically
- Solve quadratic inequalities algebraically

Concept Questions

1. What are the solutions of the inequality $x^2 + 1 < 0$?
 - a. $(-1, 1)$
 - b. $(-\infty, -1) \cup (1, \infty)$
 - c. All real numbers
 - d. No solution
2. Which points on the parabola $y = ax^2 + bx + c$ do we need to know to solve the inequality $ax^2 + bx + c > 0$?
 - a. The x -intercepts
 - b. The y -intercept
 - c. The vertex
 - d. All of these
3. Why do we need to know whether the parabola in question 2 opens up or down?
 - a. To decide whether to use $>$ or $<$ in the solution.
 - b. To decide whether the solutions lie between the x -intercepts or outside them.
 - c. To decide whether the solutions are positive or negative.
 - d. To help us find the x -intercepts.
4. What does the notation $[-3, 3]$ mean?
 - a. The point with x -coordinate -3 and y -coordinate 3 .
 - b. $x = 3$ or $x = -3$
 - c. All real numbers between -3 and 3 , excluding the endpoints.
 - d. All real numbers between -3 and 3 , including the endpoints.

Questions for Writing or Discussion

1. Explain how to use a graph to find the x -value that produces a particular y -value for a quadratic equation.
2. Explain why you cannot write the solutions to $x^2 - 4 > 0$ as a single inequality.
3. Explain how to use a graph to solve $x^2 + 4x - 77 \leq 0$.
4. Describe how to use interval notation.

Lesson 6.6 Curve Fitting

In this Lesson we use a linear system to fit a parabola through three points and use quadratic regression to fit a model to data. The Activities do not include quadratic regression.

Teaching Points for the Classroom Activities

Activity 1

- Make sure students understand that the unknowns in this system are a , b , and c .
- Point out that the easiest way to solve the system is to eliminate c first.

Activity 2

- Go over the fact that we need two points to find the equation of a line, and three points to find a parabola, unless one of the points happens to be the vertex.
- In Problem 3, tell students that they should choose points on the graph whose coordinates are easy to read.

Objectives for Lesson 6.6

- Use Gaussian elimination to fit a parabola to three points
- Use a calculator for quadratic regression
- Choose the best method to find the equation for a parabola

Concept Questions

1. Which statement is true?
 - a. A parabola is a good model for any non-linear curve.
 - b. We will get the same equation by fitting a parabola through any three points of a data set.
 - c. We can use elimination to fit a parabola through three points.
 - d. When we use quadratic regression, the lowest (or highest) point of a data set will be the vertex of the parabola.
2. How many data points must a quadratic regression graph pass through?
 - a. One
 - b. Two
 - c. Three
 - d. None
3. If you know the x -intercepts of a parabola, how many more points do you need to find its equation?
 - a. One
 - b. Two
 - c. Three
 - d. None
4. If you know the vertex of a parabola, how many more points do you need to find its equation?
 - a. One
 - b. Two
 - c. Three
 - d. None

Questions for Writing or Discussion

1. Discuss the three forms for quadratic equations, and when each is useful.
2. Explain how to fit a parabola through three points.
3. How do we choose an appropriate type of equation to fit a data set?
4. If you use the last three points in a collection of data instead of the first three points, will you get the same regression equation? Explain why or why not.

Lesson 7.1 Polynomial Functions

In this lesson we consider polynomials, and specifically cubes of binomials and factoring the sum or difference of cubes. We also look at some simple polynomial models.

Teaching Points for the Classroom Activities

Activity 1

- You may need to review the vocabulary of polynomials: degree, coefficient, descending powers, monomial, binomial, and trinomial.
- Students can usually multiply polynomials without difficulty, but you will have to show them how to find a specific term of the product. This is a good exercise for analyzing what could be just a rote manipulation.

Activity 2

- Some students have a hard time understanding how to use a formula to compute the cube of a binomial. It may help to analyze one product in detail to see how the terms arise.

- Emphasize to students that they will need to memorize the formulas for factoring the sum and difference of two cubes (as opposed to the strategies they learned for factoring quadratic trinomials.)

Activity 3

- It is helpful to physically cut the corners from a sheet of paper or cardboard and turn up the edges to illustrate the box.
- Students may not understand what is asked in part (d), so you may want to make a table showing the length, width, and height of the box for various values of x .
- Point out that (at this point) we don't have an algebraic method for solving cubic equations, so we use a graphical solution instead.

Objectives for Lesson 7.1

- Compute products of polynomials
- Compute cubes of binomials
- Factor the sum or difference of cubes
- Use polynomials as models

Concept Questions

1. The degree of a product of polynomials is the same as
 - a. the product of the degrees of the factors.
 - b. the number of terms in the product.
 - c. the sum of the degrees of the factors.
 - d. the sum of the lead coefficients.
2. What is the linear term of the product $(x - 3)(x + 5)$?
 - a. x^2
 - b. $2x - 15$
 - c. -15
 - d. $2x$
3. Which statement is true?
 - a. A cubic polynomial $y = ax^3 + bx^2 + cx + d$ describes y as a function of x .
 - b. We cannot factor the sum of two cubes.
 - c. $(2x + 3y)^3 = 8x^3 + 27y^3$
 - d. A cubic polynomial must have 4 terms.
4. How do we solve cubic equations in this class?
 - a. By factoring.
 - b. By taking cube roots.
 - c. By using the quadratic formula.
 - d. By graphing.

Questions for Writing or Discussion

1. Explain the difference between the expressions $(a + b)^3$ and $a^3 + b^3$.
2. Compare the formulas for expanding $(a - b)^2$ and $(a - b)^3$.
3. Explain why a cost function might have the shape of a cubic polynomial.
4. Explain how to find the surface area of a solid figure such as a box or a cylinder.

Lesson 7.2 Graphing Polynomial Functions

Graphing polynomials by hand is not really a useful skill any more (especially without calculus to find relative extrema), because sophisticated graphing software is readily available. This Lesson presents a more scaffolded version of Homework Problems 69 and 70, which introduce the idea of using polynomials to approximate transcendental functions. Students can use a graphing calculator or an on-line graphing tool such as Desmos to create their graphs.

Teaching Points for the Classroom Activities

Activity 1

- Students may need help with their graphing tools and entering the functions correctly.
- In part (1), ask the class what they consider a "good approximation."
- In part (3), make sure students understand why $f(x) - p(x)$ gives the error in the approximation.
- In part (4), ask students to make a table of values for $n!$ up to, say, $n = 7$.

Activity 2

- Ask the class how they can find (without graphing) the correct lead coefficient to give the desired y -intercept.

Objectives for Lesson 7.2

- Use the Factor Theorem to find the zeros of a polynomial
- Understand the short-term and long-term behavior of a polynomial
- Understand how the multiplicity of a zero affects the shape of the graph
- Sketch a rough graph of a polynomial

Concept Questions

1. At an inflection point, a graph changes
 - a. direction
 - b. concavity
 - c. sign
 - d. all of these
2. The graph of which type of polynomial must have an x -intercept?
 - a. quadratic
 - b. cubic
 - c. quartic
 - d. none of these
3. Without graphing, how can you tell that the polynomial $p(x) = (x^2 + 1)(x^2 + 2)$ has no x -intercepts?
 - a. It is quartic.
 - b. All of its coefficients are positive.
 - c. It cannot be factored further.
 - d. It has no turning points.
4. The long-term behavior of a polynomial of degree n
 - a. depends on the number of its zeros.
 - b. is the same as the behavior of $y = kx^n$.
 - c. may have a horizontal asymptote.
 - d. cannot be determined without a graph.

Questions for Writing or Discussion

1. Explain how you might estimate e^2 if you have only a four-function calculator.
2. Explain how the multiplicity of a zero affects the shape of the graph near that zero.
3. How may the graph of a polynomial differ from the graph of a power function of the same degree?
4. Describe the difference between the graphs of $y = x^3 - 2$ and $y = (x - 2)^3$.

Lesson 7.3 Complex Numbers

In this Lesson students practice graphing and performing operations on complex numbers while exploring complex roots.

Activity 1

- Begin by describing the complex plane and show students how to plot points for complex numbers. For these Activities, we will use "arrows" to locate points, so that students can more clearly see angles and rotations.
- In part (d), discuss with students how they can know that the rotation is 90° .

Activity 2

- This Activity introduces the idea that the complex roots of a number are equally spaced around the origin. After they complete part (2), you might prompt them to notice this.
- Students may have difficulty with the computations in part (3). Circulate and offer help, and if necessary demonstrate for the class.

Activity 3

- Again, students may need help with the computations.
- Offer a wrap-up stating the conclusions you hope students will draw from the Lesson.

Objectives for Lesson 7.3

- Perform operations on complex numbers
- Plot complex numbers in the complex plane
- Make observations about complex roots

Concept Questions

1. How many fourth roots does -1 have in the complex numbers?
a. 2 b. 4 c. 1 d. None
2. What is the imaginary unit?
a. -1 b. $(-1)^2$ c. i^2 d. $\sqrt{-1}$
3. What is the product of $z = a + bi$ with its complex conjugate?
a. $a - bi$ b. $a^2 - b^2$
c. $a^2 + b^2$ d. $ai + b$
4. If $z = 3 + 5i$ is a solution to a polynomial equation, then so is:
a. $5 + 3i$ b. $9 + 25$
c. $3 - 5i$ d. $5 - 3i$

Questions for Writing or Discussion

1. Explain how to divide one complex number by another.
2. What does the Fundamental Theorem of Algebra tell us? Give an example.
3. Describe how multiplying complex numbers is like multiplying binomials.
4. What can you say about the graph of a quadratic function whose zeros are complex numbers?

Lesson 7.4 Graphing Rational Functions

In this Lesson we reduce algebraic fractions and consider some simple examples using rational functions as models. We discuss the meaning of vertical and horizontal asymptotes in applications.

Teaching Points for the Classroom Activities

You may be surprised at how feeble are your students' factoring skills. If it is important to you, you might need to provide additional factoring practice. There is a Review Lesson at the end of the Activities booklet on operations with algebraic fractions.

Activity 1

- Some students may need help with parts (a) and (b). If most of the class is struggling, you may want to go over part (1) with them.
- Make sure that everyone understands part (d).
- If you led the class through part (1), you might have them tackle part (2) in their groups.

Activity 2

- Begin by reviewing the notions of revenue, profit, and cost. Make sure your students recognize the difference between price, variable cost, and average cost. Most students grasp the idea of spreading the start-up cost over the number of filters produced.
- In part (f), you may need to point out that this question does not use the formula for average cost.

Activity 3

- In part (b), help students to see that the mathematics reflects their common sense.

Objectives for Lesson 7.4

- Reduce fractions
- Use rational functions as models
- Interpret horizontal and vertical asymptotes

Concept Questions

1. Which of the following is not a rational function?

a. $y = \frac{1}{x+2}$ b. $y = 3x + 5$ c. $y = \frac{\sqrt{x^2 - 4}}{x^2 + 4}$ d. $y = \frac{2x^3 + 3x - 7}{x^4 - x^2}$

2. The fraction $\frac{x^2 - 1}{x^2 - 4}$:

- a. can be reduced to $\frac{1}{4}$.
- b. is undefined at $x = 1$ and $x = -1$.
- c. is equivalent to $\frac{(x-1)^2}{(x-2)^2}$.
- d. is equal to zero at $x = 1$ and $x = -1$.

3. Which of these functions has a horizontal asymptote at $y = 2$?

a. $y = \frac{2x+1}{x^2+1}$ b. $y = \frac{2x^2-4}{x-2}$
c. $y = \frac{2x^2+4}{x^2+1}$ d. $y = \frac{2x^2+4}{x+2}$

4. If the numerator and denominator of a rational function $y = f(x)$ are both equal to zero at $x = a$, then

- a. $f(a) = 0$
- b. $f(a) = 1$
- c. there is a vertical asymptote at $x = a$
- d. there is a hole in the graph at $x = a$

Questions for Writing or Discussion

1. Explain how to reduce an algebraic fraction, and describe some mistakes to avoid.
2. Explain carefully why $\frac{x-t}{t-x} = -1$.
3. Explain why dividing by $\frac{1}{3}$ gives the same answer as multiplying by 3.

4. Describe the shape of the graph of a rational function near a horizontal asymptote, and near a vertical asymptote.

Lesson 7.5 Equations That Include Algebraic Fractions

In this Lesson we solve equations with algebraic fractions both algebraically and graphically, and consider some applications.

Teaching Points for the Classroom Activities

Activity 1

- We review solving proportions as a special case of multiplying by the LCD.
- Part (d) illustrates a common method for estimating wildlife populations.

Activity 2

- The graphical solution helps students visualize the mathematics, but you will need to help them see how the variables appear in the graph.
- Many applications are algebraically simpler than the standard textbook problems, and hence more suitable for graphical solution.

Activity 3

- Here we use the LCD in yet a different way from the operations in previous lessons. After completing the Activity, spend some time comparing solving equations with simplifying complex fractions and adding unlike fractions.
- In Part 3, solving a formula, emphasize that we should "get the variable out of the denominator" before we can isolate it. (Students sometimes mistakenly solve for $\frac{1}{B}$, somehow thinking they can ignore the 1 in the numerator.)

Objectives for Lesson 7.5

- Solve equations involving algebraic fractions
- Check for extraneous solutions
- Solve formulas involving algebraic fractions

Concept Questions

1. A proportion is
 - a. a type of ratio.
 - b. a type of algebraic fraction.
 - c. a type of equation.
 - d. a type of word problem.
2. We can use cross-multiplying to
 - a. solve proportions.
 - b. add fractions.
 - c. multiply fractions.
 - d. do all of these.
3. When we solve an equation graphically, the solution is
 - a. the intersection point.
 - b. the x -coordinate of the intersection point.
 - c. the y -coordinate of the intersection point.
 - d. possibly extraneous.
4. Which of these is the correct first step to solve $\frac{1}{x} + \frac{1}{2x+1} = 2$?
 - a. $x + 2x + 1 = \frac{1}{2}$
 - b. $2x + 1 + x = 2$
 - c. $2x + 1 + x = 2x(2x + 1)$
 - d. $\frac{2x + 1}{x(2x + 1)} + \frac{x}{x(2x + 1)} = 2$

Questions for Writing or Discussion

1. What are extraneous solutions, and when might they occur?
2. If the variable in an equation appears in the denominator of a fraction, what should you do to solve the equation?
3. Explain how "cross-multiplying" is a special case of using an LCD.
4. Explain how to solve an equation graphically.

Lesson 8.1 Systems of Linear Equations in Two Variables

This Lesson reviews linear systems in two variables. We solve systems graphically and algebraically. Part of the purpose of this Lesson is to practice the graphing techniques presented in Chapter 1, both on paper and using the graphing calculator. We also introduce the **Intersect** feature on the calculator.

Students should learn to recognize inconsistent and dependent systems by writing both equations in the same form, either general linear or slope-intercept. For example, point out that two lines with the same slope but different y -intercepts must be parallel.

We try to avoid artificial applications such as coin and age problems, considering instead weighted averages, which include mixture and interest problems. Finding the equilibrium price for supply and demand equations, and the break-even point from cost and revenue appear in the Homework problems.

Teaching Points for the Classroom Activities

Activity 1

- Part (c): After practicing the intercept method of graphing, many students have forgotten that we can also graph a line by plotting points.
- Part (e): Explain that the intersection point is the point at which $S = E$, so by setting the expressions for S and E equal and solving the equation, we are finding the value of t that makes S and E equal, that is, the month in which the total operating costs are equal.

Activity 2

- Make sure students write down exactly what x and y represent.
- Part (d) is difficult for students. Suggest that they read the problem again, thinking about what x and y represent.

Activity 3

- Use part (a) to review the two graphing methods: the intercept method and the slope-intercept method. Ask students which method is more efficient on each equation.

Activity 4

- Explain how to add or subtract the speed of the wind (or the current in a river) to the speed of the airplane.
- Point out explicitly that x and y are NOT the entries in the Rate column.

Activity 5

- Many students still believe that we can add percents, even of different amounts. Try this example: If you have two jars of red and blue marbles, each of which is 50% red marbles, and you pour both jars into a larger jar, will the result be 100% red marbles?
- Point out that mixture problems are actually easy: after we fill in the first two columns of the table with information from the problem, we multiply across to get the third column and add down columns 1 and 3 to get the equations.

Objectives for Lesson 8.1

- Verify a solution of a system
- Solve a linear system graphically
- Solve systems using substitution and elimination
- Identify inconsistent and dependent systems
- Model and solve applied problems with a system of equations

Concept Questions

1. Which points on the graphs satisfy both equations in a system?
 - a. Their x -intercepts
 - b. Their y -intercepts
 - c. Their intersection points
 - d. None of the points
2. How can you verify that a given point is the solution of a system?
 - a. Add the equations together.
 - b. Show that it makes one of the equations true.
 - c. Show that it makes both equations true.
 - d. Subtract the equations.
3. If a system is inconsistent, the two lines:
 - a. are parallel.
 - b. are perpendicular.
 - c. intersect.
 - d. have the same graph.
4. What is an equilibrium price?
 - a. A fair and balanced price
 - b. The price that yields maximum profit
 - c. The price at which supply equals demand
 - d. The price at which revenue equals costs

Questions for Writing or Discussion

1. What do the points on the graph of an equation represent?
2. Is it possible for a linear system to have more than one solution? Why or why not?
3. When is the substitution method easier to use than the elimination method. Why?
4. Explain how to use linear combinations to identify inconsistent and dependent systems.

Lesson 8.2 Systems of Linear Equations in Three Variables

This Lesson extends the elimination method to 3×3 linear systems. It is probably too ambitious to expect students to understand the geometric interpretation as planes in space, or the various configurations of dependent systems. We concentrate on the process of Gaussian elimination, because this useful technique generalizes to higher order linear systems.

Students usually have no trouble mastering back-substitution in a triangular system. Although there is a lot of grumbling initially at the number of steps in Gaussian reduction, if you show students how to keep their work organized, after the second example they feel more comfortable.

Teaching Points for the Classroom Activities

Activity 1

- Let students try this on their own in groups. Even if they didn't read the Lesson, they can usually figure it out.

Activity 2

- You might want to work out an Example for students before asking them to try Activity 2. Model the process for them as shown in the Activity. Colored pens help.
- Point out that clearly showing the steps is part of the problem: if you can't follow their work, you can't grade it.

Activity 3

- Students can be flummoxed if one or more of the equations is "missing" a variable. Show them how this actually shortens their work by eliminating Step 2 of the solution process, if they choose their equations carefully.

Objectives for Lesson 8.2

- Use back-substitution to solve a triangular system
- Use Gaussian reduction to solve a 3×3 linear system
- Model a problem with three linear equations in three unknowns

Concept Questions

1. What sort of system can be solved by back-substitution?
 - a. Any linear system
 - b. A Gaussian system
 - c. A triangular system
 - d. A dependent system
2. After we eliminate one variable from a pair of equations, what is the next step?
 - a. Eliminate a different variable from the same pair of equations.
 - b. Eliminate the same variable from a different pair of equations.
 - c. Eliminate a different variable from a different pair of equations.
 - d. Substitute into the third equation.
3. What does it mean if you obtain an equation of the form $0x + 0y + 0z = k$, $k \neq 0$?
 - a. The system has no solution.
 - b. The system is dependent.
 - c. The solution is $(0, 0, 0)$.
 - d. The system is triangular.
4. What should you do if one of the original equations in a 3×3 system is "missing" one of the variables?
 - a. Throw out that equation.
 - b. Add a variable to both sides of the equation.
 - c. Eliminate that variable from the other two equations.
 - d. Nothing: there is no solution.

Questions for Writing or Discussion

1. What is a solution to a 3×3 linear system?
2. Why do we try to reduce a 3×3 linear system to triangular form?
3. Explain how you know immediately that the following system has no solution:

$$\begin{aligned}x + y + z &= 4 \\2x + 2y + 2z &= 5 \\x - y - 3z &= 6\end{aligned}$$

4. Explain how you know immediately that the following system is dependent:

$$\begin{aligned}x + y + z &= 3 \\2x + 2y + 2z &= 6 \\-x - y - z &= -3\end{aligned}$$

Lesson 8.3 Solving Linear Systems Using Matrices

In this Lesson we solve 2×2 and 3×3 linear systems using matrices.

Teaching Points for the Classroom Activities

Activity 1

- There is not much advantage to using a matrix for a 2×2 system. Point out that we are learning the steps to use on a larger system.
- Point out that reduced row echelon form produces a diagonal coefficient matrix.

Activity 2

- This is a very easy example of reducing a 3×3 matrix. If you have time to do another and your students are ready for it, you could choose a harder one from the Homework Problems.
- You might ask students whether they prefer the upper triangular or the reduced row echelon form for solving a system.

Concept Questions

1. Which of the following is not an allowed row operation?
 - a. Divide the entries in a row by a nonzero number
 - b. Add two rows together
 - c. Interchange two elements in a row
 - d. Interchange the first and last rows
2. Why is reduced row echelon form useful?
 - a. It makes the elements in each row smaller.
 - b. It lets us read the solutions of a system.
 - c. It reduces the number of rows.
 - d. It solves the matrix.
3. What happens if we cannot obtain nonzero entries on the diagonal?
 - a. There is no solution.
 - b. There is no unique solution.
 - c. We should reduce the matrix.
 - d. We should augment the matrix.
4. What should you do if the first entry in the first row is zero?
 - a. Interchange two columns
 - b. Interchange two rows
 - c. Add the first row to the second row
 - d. Obtain zeros below that entry

Questions for Writing or Discussion

1. What is the difference between a coefficient matrix and an augmented matrix?
2. How is matrix reduction similar to Gaussian reduction?
3. What is the order of a matrix?
4. Why do we want to put a matrix into upper triangular form?

Lesson 8.4 Linear Inequalities

In this Lesson we graph the solutions to linear inequalities in two variables, and to systems of linear inequalities. We also find the vertices of the solution set.

Teaching Points for the Classroom Activities

Activity 1

- If necessary, begin by reviewing how to graph the solution set of a single linear inequality. Try to make sure students understand that the solutions all lie on one side of the boundary line.
- Stress that it's important to complete the graph of each inequality in the system separately, before going on to the next one. Have them number the inequalities and label each boundary line with its number.
- Some students enjoy using colored pencils for their solution sets. Otherwise, make sure they shade each solution set lightly and in a different direction, so that they can see the intersection.

Activity 2

- You may be surprised that many students cannot tell you how to find the intersection point of two lines.
- If students numbered the boundary lines in Activity 1, now they can identify quickly which pair of equations yields each vertex.

Activity 3

- This is a long application, and might be appropriate for a project or group homework assignment.

Objectives for Lesson 8.4

- Graph the solution set for an inequality in two variables
- Graph the solution set for a system of inequalities
- Find the vertices of the solution set

Concept Questions

1. The solutions of $x > -4$ all lie to which side of the boundary line?
 - a. Above
 - b. Below
 - c. To the left
 - d. To the right
2. Which point would not be a good test point for the inequality $y \leq \frac{2x}{3}$?
 - a. $(0, 0)$
 - b. $(1, 1)$
 - c. $(2, 3)$
 - d. $(-1, -1)$
3. To find the vertices of the solution set for a system of inequalities, we
 - a. use a test point.
 - b. use the vertex formula.
 - c. solve a system of equations.
 - d. find the x -intercepts.
4. The first quadrant includes all the solutions of
 - a. $x + y > 0$
 - b. $x - y > 0$
 - c. $xy > 0$
 - d. $x > 0, y > 0$

Questions for Writing or Discussion

1. What is the difference between the graphs of an equation in two variables and an inequality in two variables?
2. When is it wrong to use $(0, 0)$ as a test point? What should you do instead?
3. Why do we shade the intersection of the solution sets when solving a system of inequalities?
4. Give three different ways we have used the word "vertex" in this course, and provide a sketch illustrating each meaning.

Lesson 8.5 Linear Programming

This Lesson guides students through the steps of a linear programming problem.

Activity 1

- It doesn't matter which variable is x and which is y , but once students have identified the two variables, you might ask them all to make the same choice, so that everyone's graph is the same.
- Note that the third inequality in part (1b) involves y only.
- In part (1c), the first two boundary lines should be easy to graph by the intercept method.
- Tell students to shade the solutions for each inequality lightly, so that they can see their intersection clearly.
- In part (2d), you may have to remind students how to find the intersection point of two lines.
- In part (3), it may be helpful to choose one or two random points in the set of feasible solutions and evaluate the objective function at each, and then ask students to say what each means.
- In your wrap-up, include the fact that the optimum solution always occurs at one of the vertices of the set of feasible solutions.

Activity 2

- This is the same problem that appears in the Projects for Chapter 8, with a little more scaffolding. If your students need a greater challenge, you might assign this as an outside class group project.

Objectives for Lesson 8.5

- Write the objective function and constraints for a linear programming problem
- Graph the set of feasible solutions
- Find the optimal solution

Concept Questions

1. Which of these could be an objective function?
 - a. $2x + 5y \leq 50$
 - b. $M = 2x + 5y$
 - c. $y \geq 2x + 50$
 - d. $2x + 5y = 50$
2. What is the set of feasible solutions?
 - a. The vertices of the solution set
 - b. The values of the objective function
 - c. The set of points that satisfy the constraints
 - d. The boundary lines of the inequalities
3. Linear programming is used to
 - a. find the vertices of a region.
 - b. solve a system of inequalities.
 - c. write a set of inequalities in two variables.
 - d. find an optimal solution subject to a set of constraints.
4. What is an optimal solution?
 - a. The largest or smallest feasible solution
 - b. The largest or smallest value of the objective function in the set of feasible solutions
 - c. The vertex farthest from the origin
 - d. The largest or smallest constrained y -value

Questions for Writing or Discussion

1. Explain how to find the set of feasible solutions.
2. What fact helps us find the optimal solution?
3. In Activity 1, why wouldn't Tanya produce only mugs, which have a higher selling price?
4. How is the procedure for linear programming different if you are looking for a minimum value of the objective function, instead of a maximum value?